Atomic functions in nonparametric estimations of probability density functions and their derivatives

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Nonparametric estimations of probability density function (PDF) [1–3] and its derivatives are considered. New constructions of weight functions (WF) [4, 5] with the compact support are offered and proved, allowing to build admissible estimations, both the density of probability and its derivatives 1st and 2nd orders. The numerical experiment and the physical analysis of its results confirm efficiency of nonparametric characteristics estimation of stochastic process.

1 Introduction

The urgency of application of nonparametric estimations [6, 7] in physical applications is caused by simplicity of structure and possibility of their use when the restored value is unknown. Therefore the new mathematical apparatus of nonparametric statistics received by means of theory of AF will allow to estimate characteristics of investigated sequences without having the aprioristic parametrical information.

2 Constructions of admissible weight functions

Let $X_1, X_2, ..., X_n$ is sequence of n samples of random variable independent observations with unknown PDF $f(x)$. Its nonparametric estimation is defined by

$$f_n(x) = \frac{1}{nh}\sum_{j=1}^{n} K\left(\frac{X_j - x}{h}\right), \tag{1}$$

where $h = h(n)$ is some sequence of positive numbers, $\lim\limits_{n\to\infty} h(n) = 0$ and $K(x)$ is even function satisfying to normalization requirement

$$\int_{-\infty}^{\infty} K(x)dx = 1, \quad K(x) \in L_2.$$

Weight function is called 'admissible' if its Fourier transform (FT) is nonnegative and no more than 1 for all real frequencies. Quality criteria of estimation of $f_n(x)$ is an mean integrated square error. If WF is admissible then mean integrated square error of estimation received with its help can't be reduced simultaneously for all PDFs. Therefore will consider of admissible WF construction on example AF $h_a(x)$ [4]. The FT of AF $h_a(x)$ is

$$\phi_a(\omega) = \prod_{k=1}^{\infty} \frac{\sin\left(\omega/a^k\right)}{\omega/a^k}.$$

Let's enter the function

$$ch_a(x) = h_a(x) * h_a(x).$$

Repeating operation of convolution of $(l-1)$ times will make function $ch_{a,l}(x)$. It is obvious that

$$\widehat{ch_{a,l-1}}(\omega) = \left(\prod_{k=1}^{\infty} \frac{\sin\left(\omega/a^k\right)}{\omega/a^k}\right)^l, \tag{2}$$

$$\operatorname{supp}\left(ch_{a,l}(x)\right) = \left[-\frac{l}{(a-1)}, \frac{l}{(a-1)}\right],$$

$$\int_{-l/(a-1)}^{l/(a-1)} ch_{a,l}(x)dx = 1, \quad l = 1, 2, ...$$

Admissible WF will construct by means of spectrum expression

$$\Psi_r(\omega) = 1 - \left(1 - \phi_a(\omega)\right)^{r/2}, \quad r = 2, 4, ...,$$

which positively and also no more than 1. Thus, expression for calculation admissible WF is the following:

$$K_{a,r}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_r(\omega) e^{i\omega x} d\omega. \tag{3}$$

Here $r = 2, 4, ...$ For example

$$K_{a,2}(x) = ch_{a,1}(x), K_{a,4}(x) = 2ch_{a,1}(x) - ch_{a,2}(x),$$
$$K_{a,6}(x) = 3ch_{a,1}(x) - 3ch_{a,2}(x) + ch_{a,3}(x).$$

Admissible value estimation of $f'(x)$ will receive in the following form:

$$Df_n(x) = \frac{1}{nh^2}\sum_{j=1}^{n} N\left(\frac{X_j - x}{h}\right), \tag{4}$$

$$N_r(x) = -\frac{dK_{r-1}(x)}{dx},\ r = 3, 5, \ldots.$$

Similarly for the second derivative of PDF

$$D^2 f_n(x) = \frac{1}{nh^3} \sum_{j=1}^{n} M\left(\frac{X_j - x}{h}\right), \qquad (5)$$

$$M_r(x) = -\frac{d^2 K_{r-2}(x)}{dx^2},\ r = 3, 5, \ldots.$$

Constructed WF have compact support and they are infinitely differentiable. The view of weight functions $K_{a,r}(x)$ and also their derivatives are presented on Fig. 1.

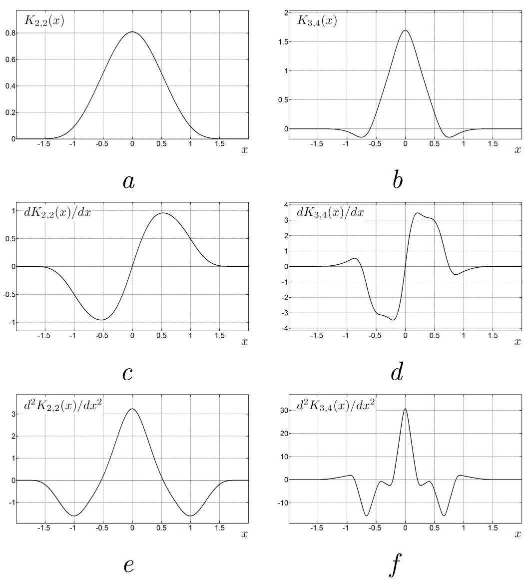

Figure 1: View of WF $K_{a,r}(x)$ and also their derivatives for (a, c, e): $a = 2, r = 2$ and (b, d, f): $a = 3, r = 4$

3 Physical properties of analytical two-dimensional Kravchenko–Rvachev wavelets

For research of WF we will use the following modified physical characteristics: width of spectral density function (SDF) on level *-3 dB* (γ_3). Relative width of SDF on level *-6 dB* $\left(\gamma_{4/3} = \gamma_4/\gamma_3\right)$. Maximum level of the SDF side-lobes (in dB) $\gamma_9 = 20 \lg \max_k \left|\frac{\hat{y}(\omega_{m,k})}{\hat{y}(\omega_{max})}\right|$, where $\omega_{m,k}$ is ordinate of k-th SDF local maximum differ main maximum ω_{max}. L_2-norm of WF (P). Uncertainty constants on time and frequency that is characterized time-frequency localization $\Delta = \Delta_x \Delta_\omega$, where

$$x^* = \frac{1}{\|y(x)\|^2} \int_{-\infty}^{\infty} x \left|y(x)\right|^2 dx,$$

$$\Delta_x = \frac{1}{\|y(x)\|} \left(\int_{-\infty}^{\infty} (x - x^*)^2 \left|y(x)\right|^2 dx\right)^{1/2},$$

$$\omega^* = \frac{1}{\|\hat{y}(\omega)\|^2} \int_{-\infty}^{\infty} \omega \left|\hat{y}(\omega)\right|^2 df_1 df_2,$$

$$\Delta_\omega = \frac{1}{\|\hat{y}(\omega)\|} \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (\omega - \omega^*)^2 \left|\hat{y}(\omega)\right|^2 d\omega\right)^{1/2}.$$

Support of WF (supp), and also effective support $\mathrm{supp}_E = \left\{x \left| \|f(x)\|_{L_2} = 0.999P\right.\right\}$. In the Table 1 physical characteristics of WF for various a and r are resulted.

Table 1: Physical characteristics of $K_{a,r}(x)$ for different a and r.

a	2	3	4	5
		$r = 2$		
γ_3	1.720	2.693	3.815	4.787
$\gamma_{4/3}$	1.391	1.444	1.412	1.375
γ_9	-46.59	-34.03	-30.49	-28.99
P	0.945	0.735	0.622	0.549
Δ	0.502	0.508	0.513	0.518
supp	4	2	4/3	1
supp_E	2.029	1.233	0.898	0.708
		$r = 4$		
γ_3	2.581	4.114	5.610	7.181
$\gamma_{4/3}$	1.232	1.236	1.240	1.229
γ_9	-39.94	-27.99	-23.94	-22.98
P	0.883	0.682	0.573	0.503
Δ	0.544	0.560	0.574	0.585
supp	8	4	8/3	2
supp_E	2.669	1.662	1.218	0.964

4 Numerical experiment

Let's consider an example of probability density estimation of random values X_n sequence with the normal law of distribution

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where mathematical expectation $\mu = 0.25$, standard deviation $\sigma = 0.5$ (see Fig. 2a). On Fig. 2b-d the results of PDF estimation $f_n(x)$, and also its first two derivatives for $a = 2$, $r = 2$ and $n = 2500$

are presented. Let's estimate of characteristics of the distribution calculated on sequence, and also from estimations. On Fig. 3a,b the schedules of $(\mu^* - \mu_n)$ and $(\sigma^* - \sigma_n)$ are resulted. Here $\mu^* = x^*, \sigma^* = \Delta_x$ if $y(x) = f_n(x)$. Apparently from

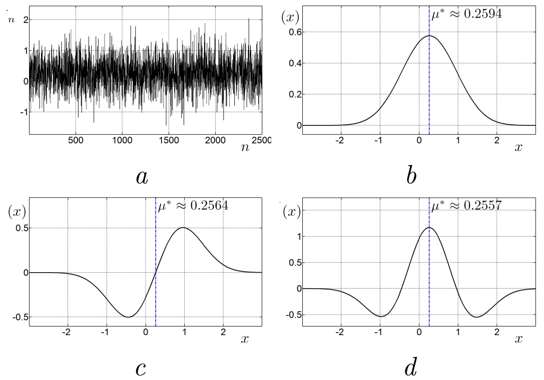

Figure 2: (a): Random values X_n sequence, $n = 2500$, $(b–d)$: estimation of $f_n\,(x)$, $Df_n\,(x)$ $D^2 f_n\,(x)$, $a = 2$, $r = 2$

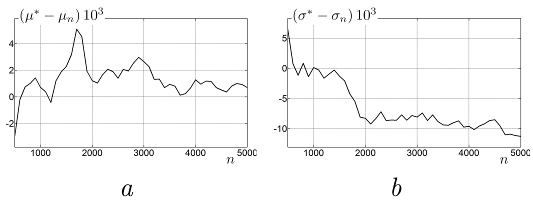

Figure 3: Dependence of (a): $(\mu^* - \mu_n)$ and (b): $(\sigma^* - \sigma_n)$ from increasing of n $(a = 2, r = 2)$

plots at increase of WF order, we receive more smooth PDF estimation and at decrease more exact calculation of mathematical expectation. The Fig. 4 illustrates change of $(\sigma^* - \sigma_n)$ in depending on parameters a and r. At increase a or r more exact calculation σ^* is observed.

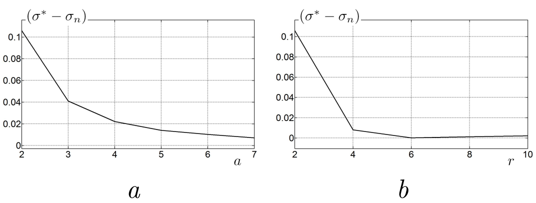

Figure 4: Dependence $(\sigma^* - \sigma_n)$ from (a): a $(r = 2)$ and (b): r $(a = 2)$, $n = 2500$

5 Conclusion

Thus, on the basis of the theory of AF the new construntions of WS with the compact support are offered. Admissible nonparametric probability density function estimations and its derivatives of 1st and 2nd orders of random variables sequences are constructed. Presence of function (3) parameters allows to expand applicability of the offered estimations to stochastic processes of the various physical nature. The carried out physical analysis confirms efficiency of the new nonparametric probability density function estimations.

References

[1] V.A. Vasiliev, A.V. Dobrovidov, G.M. Koshkin, 2004, Nonparametric Estimation of Functionals of Stationary Sequences Distributions, *Moscow, Nauka.*

[2] W. Hardle, 1994, Humboldt-Universitat, Berlin, *Moscow, Nauka.*

[3] L. Devroye, L. Gyorfi, 1985, Nonparametric Density Estimation: The L_1 View, *New York, John Wiley.*

[4] V.F. Kravchenko, 2003, Lectures on the Theory of Atomic Functions and Their Some Applications *Moscow, Publishing House Radio Engineering.*

[5] Edited by V.F. Kravchenko, 2007, Digital Signal and Image Processing in Radio Physical Applications, *Fizmatlit, Moscow.*

[6] V.F. Kravchenko, H.M. Perez-Meana, V.I. Ponomaryov, 2009, Adaptive Digital Processing of Multidimentional Signals with Applications, *Fizmatlit, Moscow.*

[7] V.F. Kravchenko, O.V. Kravchenko, A.R. Safin, 2009, Atomic Functions in Probability Theory and Stochastic Processes, *Successes of Modern Radio Electronics*, no. **5**, pp. 23–38.

[8] V.F. Kravchenko, O.V. Kravchenko, A.R. Safin, D.V. Churikov, 2009, New class of probability weight functions in digital signal and image processing, *An International Journal Electromagnetic Waves and Electronic Systems*, vol. **14**, no. **9**, pp. 31–44.