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REVIEW

Atomic, WA-Systems, and R-Functions Applied in Modern Radio Physics Problems: Part I

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Abstract—The main physical applications of atomic, WA-systems, and R-functions are presented. The Whittaker–Kotelnikov–Shannon sampling theorem is generalized on the basis of atomic functions. Applications of atomic functions in the theory of probability and random processes, interpolation of stationary random processes with atomic functions, and a new class of probabilistic weighting functions used in digital signal and image processing are considered.

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INTRODUCTION

For over 40 years, the theory of R- and atomic functions (AFs) [1–45] and, recently, the theory of WA-system functions [8–19] have been actively developed in various physical applications. The theories of atomic and R-functions are elaborated in monographs [1–13]. In Russia and foreign countries, intense investigations oriented to the development and application of the theory of AFs combined with Rvachev functions (R-functions) in physics and engineering are carried out for a wide frequency spectrum extending from the microwave to optical range. Being vast, the review is divided into four parts. In this first part the following issues are considered:

(i) the main physical applications of atomic, WA-systems, and R-functions,

(ii) the Whittaker–Kotelnikov–Shannon (WKS) sampling theorem and its generalization based on AFs,

(iii) atomic functions in the theory of probability and random processes,

(iv) interpolation of stationary random processes with AFs,

(v) a new class of probabilistic weighting functions (WFs) used in digital signal and image processing.

1. THE MAIN PHYSICAL APPLICATIONS OF ATOMIC, WA-SYSTEMS, AND R-FUNCTIONS

The recent investigations in the field of AF applications include the following scientific lines:

(i) generalized Kotelnikov series based on AFs $h_a(x)$ and $fup_N(x)$,

(ii) the AF-based generalized N-dimensional WKS theorem,

(iii) AF-based Levitan and Strang–Fix polynomials,

(iv) WA-systems of Kravchenko–Rvachev functions and their application for detecting short-term alternating-sign and ultrawideband (UWB) physical processes,

(v) AF-based spectral processing of UWB signals,

(vi) Kravchenko–Kotelnikov analytical wavelets in digital signal processing (DSP),

(vii) Kravchenko–Pustovoit time WFs (windows) in surface-acoustic-wave DSP devices,

(viii) Kravchenko–Kotelnikov WFs in digital signal spectroscopy,

(ix) a Cohen's class time–frequency distribution with AFs in a nonlinear DSP,

(x) atomic and atomic-fractal functions in the antenna theory,

(xi) investigation of the behavior of an electric-dipole's field pulse,

(xii) DSP in synthesized aperture radar,

Table 1. Basic parameters of the new Kravchenko and classical windows

WFs (windows)	ENB	OC, %	SAM, dB	TL_{max}, dB	SLL_{max}, dB	CG
K_2^4	1.9861	4.2498	0.8518	3.8318	–51.6112	0.3610
$K_2^2G_2$	1.8105	7.4054	1.0259	3.6038	–53.7964	0.3944
K_2G_3	1.9643	4.7297	0.8781	3.8101	–68.8390	0.3614
$K_4^2G_2^2$	1.9631	4.7869	0.8809	3.8103	–70.6203	0.3607
$K_4^4G_2$	1.9696	4.6700	0.8742	3.8180	–71.2806	0.3598
$K_4^4G_3$	2.0415	3.7429	0.8156	3.9152	–74.8054	0.3467
$K_4G_{3.5}$	1.8007	7.3910	1.0249	3.5793	–74.9523	0.3988
Rectangular	1.0000	50.000	3.9224	3.9224	–13.2799	1.0000
Triangular	1.3333	25.0001	1.8242	3.0736	–26.5077	0.5000
Gaussian $(\alpha = 3.5)$	1.9765	4.6147	0.8702	3.8292	–71.0006	0.3579
Hamming	1.3638	23.3241	1.7492	3.0967	–45.9347	0.5395
Blackman–Harris (four-term)	2.0044	3.7602	0.8256	3.8453	–92.0271	0.3587
Nuttall (four-term)	1.9761	4.1760	0.8506	3.8087	–97.8587	0.3636
Dolph–Chebyshev $(\alpha = 3.5)$	1.6328	11.8490	1.2344	3.3636	–70.0161	0.4434
Bernstein–Rogozinski	1.2337	31.8309	2.0982	3.0103	–23.0101	0.6366
Kaiser $(\alpha = 3)$	1.7952	7.3534	1.0226	3.5639	–69.6568	0.4025

The following notation is introduced: (K) Kravchenko, (G) Gauss, (ENB) equivalent noise bandwidth, (OC) overlapping correlation (for 50% overlap), (SAM) spurious amplitude modulations, (TL_{max}) maximum transform loss, (SLL_{max}) maximum sidelobe level, (CG) coherent gain.

(xiii) orthogonal wavelet bases in signal and image digital processing,

(xiv) AFs in the theory of probability and random processes,

(xv) synthesis of 2D digital filters of a complex geometry,

(xvi) multidimensional filtering,

(xvii) nonparametric signal estimation,

(xviii) gyroscopy,

(xix) construction of Kravchenko–Kotelnikov–Gauss and Kravchenko–Kotelnikov–Levitan–Gauss WFs,

(xx) a new class of wavelets based on AF $h_a(x)$,

(xxi) a new class of Kravchenko–Rvachev analytical wavelets,

(xxii) AF and wavelet-based digital processing and spectral estimation of UWB signals,

(xxiii) Kravchenko and Kravchenko–Rvachev WFs in problems of radar image construction and antenna aperture synthesis,

(xxiv) AFs in the theory of probability and stochastic processes,

(xxv) AFs in problems of physical electronics,

(xxvi) the theory of R-functions and WA-systems functions applied for the solution of boundary value problems of mathematical physics.

Spectral Properties of Atomic Functions in Digital Signal Processing

As is known, the choice of WFs [4–21] is one of the main points in classical problems of spectral signal estimation. Window DSP, which, in particular, is applied to control physical parameters, is based on the presence of sidelobes in spectral estimates. The ideas and results previously presented are used to develop new approaches to spectral estimation. Thus, new WFs are constructed in the form of a direct product of AFs fup_N and classical Gaussian, Bernstein, and Dolph–Chebyshev windows (Table 1). These results are the basis of digital spectral processing of multidimensional signals, antenna aperture synthesis, the solution of problems of signal compression, computer tomography and thermography, and medical diagnostics.

2. THE WHITTAKER–KOTELNIKOV–SHANNON SAMPLING THEOREM AND ITS GENERALIZATION ON THE BASIS OF ATOMIC FUNCTIONS

It is known that transmission of various signals over communication systems usually involves time functions whose spectrum is bounded, i.e., contains no frequencies above a certain boundary value. Such functions exhibit unique properties, which were first discovered by Kotelnikov in 1933. He formulated these properties in the theorem [46] playing a fundamental role in the communication and informatics theory and various other physical applications. Signals with a finite spectrum can also be interpolated with the use of the Fourier transforms (FTs) of AFs [5–9, 14–19], because the zeros of these transforms are regularly arranged. In addition, at infinity, the spectra of AFs approach zero much faster than functions sinc(x), a circumstance that allows truncating the interpolation series to a comparatively small number of terms.

A. The Generalized Kravchenko–Kotelnikov Sampling Theorem for the 1D Case

Signal $s(t)$ with finite bandwidth $\hat{s}(\omega)$ can be represented in the form [5, 16, 22–29]

$$s(t) = \sum_{k=-\infty}^{\infty} s(k\Delta) \prod_{j=1}^{\infty} \operatorname{sinc}\left(\frac{\pi}{\Delta a^{j-1}}(t - k\Delta)\right). \tag{1a}$$

This expression satisfies all the conditions of the Kotelnikov theorem and exhibits a better convergence, especially, when signals local in time and discontinuous signals are restored. In the calculation it is necessary to retain a finite number of terms in the product on the right-hand side of series (1a). In this case the exact expansion is valid:

$$s(t) = \sum_{k=-\infty}^{\infty} s(k\Delta) \prod_{j=1}^{M} \operatorname{sinc}\left(\frac{\pi}{\Delta a^{j-1}}(t - k\Delta)\right), \tag{1b}$$

$$a\left(1 + a^{-M}\right) > 2, \quad \Delta = \frac{\pi}{\Omega}\frac{a\left(1 + a^{-M}\right) - 2}{a - 1}.$$

The minimum values of a can be found from the solution of the transcendental equation $a\left(1 + a^{-M}\right) = 2$. The Kotelnikov series follows from generalized series (1b) when $M \to \infty$ and series (1a) follows from (1b) at the limit as $M = 1$. Let us formulate the N-dimensional WKS theorem. Assume that N-dimensional signal $f(x) = f(x_1,\ldots,x_N) \in L_2\left(\mathbf{R}^N\right)$ is specified and that its FT $F(\omega) = \mathfrak{F}[f(x)](\omega)$ satisfies the condition of the spectrum boundedness

$$F(\omega) = \begin{cases} F(\omega) & \text{when } \omega \in A, \\ 0 & \text{when } \omega \notin A. \end{cases}$$

Here, A is a certain domain in $\mathbf{R}^N$. Let $x_\Delta = (x_{1\Delta},\ldots,x_{N\Delta}) \in \mathbf{R}^N$ denote the spatial discretization step and $\omega_\Delta = (\omega_{1\Delta},\ldots,\omega_{N\Delta}) = \left(\frac{\pi}{x_{1\Delta}},\ldots,\frac{\pi}{x_{N\Delta}}\right) \in \mathbf{R}^N$ denote the discretization frequency. We introduce the N-dimensional rectangular domain

$$D = [-\omega_{1\Delta}, \omega_{1\Delta}] \times \cdots \times [-\omega_{N\Delta}, \omega_{N\Delta}].$$

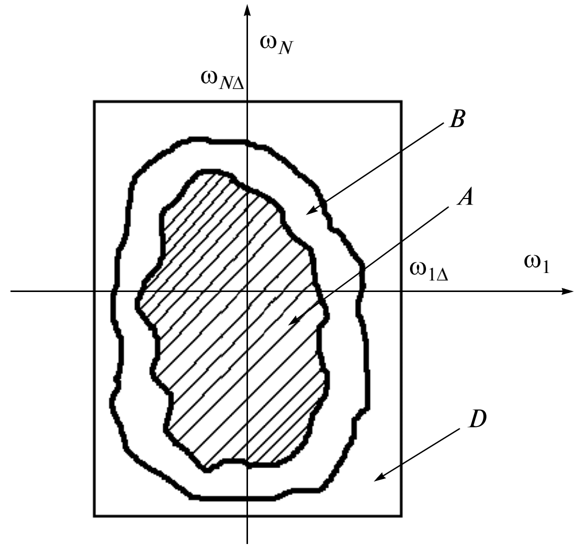

Fig. 1. Reference frequency domains A, B, and D.

Theorem 1 (The *N*-dimensional sampling Theorem). Let the signal spectrum be bounded and located in domain $A \subset \mathbf{R}^N$. Then, when the domain entirely covers domain $A(D \supseteq A)$, the complete spectrum can be restored from the set of samples with the help of the following formula:

$$\begin{aligned} f(x_1,\ldots,x_N) = \sum_{l_1,\ldots,l_N \in \mathbf{Z}} (x_{1\Delta}\ldots x_{N\Delta}) f(l_1 x_{1\Delta},\ldots,l_N x_{N\Delta}) \\ \times g_B(x_1 - l_1 x_{1\Delta},\ldots,x_N - l_N x_{N\Delta}), \end{aligned} \tag{2}$$

where $\mathbf{Z}$ is the set of integers,

$$g_B(x_1,\ldots,x_N) = \frac{1}{(2\pi)^N}$$

$$\times \int\ldots\iint_B \exp(i(x_1\omega_1 + \ldots + x_N\omega_N))\, d\omega_1\ldots d\omega_N.$$

Here, $B \subset D$ and $A \subseteq B$.

Proof. Let us form a spectrum that is a function periodically extended with a frequency domain period (Fig. 1). Then the spectrum that is a periodic function can be expanded in the Fourier series

$$\begin{aligned} &\hat{F}(\omega) \\ &= \sum_{l_1,\ldots,l_N \in \mathbf{Z}} \alpha_{l_1,\ldots,l_N} \exp\left(-i\pi\left(l_1 \frac{\omega_1}{\omega_{1\Delta}} + \ldots + l_N \frac{\omega_N}{\omega_{N\Delta}}\right)\right) \\ &= \sum_{l_1,\ldots,l_N \in \mathbf{Z}} \alpha_{l_1,\ldots,l_N} \exp(-i(l_1\omega_1 x_{1\Delta} + \ldots + l_N\omega_N x_{N\Delta})), \end{aligned} \tag{3}$$

$$\alpha_{l_1,\ldots,l_N} = \frac{1}{2\omega_{1\Delta}\ldots 2\omega_{N\Delta}} \int_{-\omega_{1\Delta}}^{\omega_{1\Delta}} \ldots \int_{-\omega_{N\Delta}}^{\omega_{N\Delta}} \hat{F}(\omega_1,\ldots,\omega_N)$$
$$\times \exp\left(i\left(l_1\omega_1 x_{1\Delta} + \ldots + l_N\omega_N x_{N\Delta}\right)\right) d\omega_1\ldots d\omega_N$$
$$= \frac{x_{1\Delta}\ldots x_{N\Delta}}{(2\pi)^N} \int\ldots\int_A F(\omega_1,\ldots,\omega_N)$$
$$\times \exp\left(i\left(l_1\omega_1 x_{1\Delta} + \ldots + l_N\omega_N x_{N\Delta}\right)\right) d\omega_1\ldots d\omega_N$$
$$= (x_{1\Delta}\ldots x_{N\Delta}) f(l_1 x_{1\Delta},\ldots,l_N x_{N\Delta}).$$

The substitution of the expression for $\alpha_{l_1,\ldots,l_N}$ into (3) yields

$$\hat{F}(\omega_1,\ldots,\omega_N) = \sum_{l_1,\ldots,l_N\in\mathbf{Z}} x_{1\Delta}\ldots x_{N\Delta} \times f(l_1 x_{1\Delta},\ldots,l_N x_{N\Delta}) \times \exp\left(-i\left(l_1\omega_1 x_{1\Delta} + \ldots + l_N\omega_N x_{N\Delta}\right)\right). \quad (4)$$

Consider a nonperiodic signal spectrum that can be obtained from (3) by transmitting the signal through an ideal lowpass filter with the frequency characteristic (FC)

$$p_B(\omega) = \begin{cases} 1 \text{ when } \omega \in B, \\ 0 \text{ when } \omega \notin B, \end{cases}$$

where B is a certain domain in $\mathbf{R}^N$. Then,

$$F(\omega_1,\ldots,\omega_N) = \hat{F}(\omega_1,\ldots,\omega_N) p_B(\omega)$$
$$= \sum_{l_1,\ldots,l_N\in\mathbf{Z}} (x_{1\Delta}\ldots x_{N\Delta}) f(l_1 x_{1\Delta},\ldots,l_N x_{N\Delta}) \quad (5)$$
$$\times \exp\left(-i\left(l_1\omega_1 x_{1\Delta} + \ldots + l_N\omega_N x_{N\Delta}\right)\right) p_B(\omega).$$

The inverse FT of (5) yields (2). As is known the WKS theorem characterizes the limit capabilities of a communication channel. Then series (2) should be understood as

$$f(x_1,\ldots,x_N) = \lim_{L_1,\ldots,L_N\to\infty} \sum_{l_1=-L_1}^{L_1} \ldots \sum_{l_N=-L_N}^{L_N} x_{1\Delta}\ldots x_{N\Delta}$$
$$\times f(l_1 x_{1\Delta},\ldots,l_N x_{N\Delta})\, g_B(x_1 - l_1 x_{1\Delta},\ldots,x_N - l_N x_{N\Delta}),$$

$$f^{L_1,\ldots,L_N}(x_1,\ldots,x_N) = \sum_{l_1=-L_1}^{L_1} \ldots \sum_{l_N=-L_N}^{L_N} x_{1\Delta}\ldots x_{N\Delta}$$
$$\times f(l_1 x_{1\Delta},\ldots,l_N x_{N\Delta})\, g_B(x_1 - l_1 x_{1\Delta},\ldots,x_N - l_N x_{N\Delta}).$$

Then the convergence is determined in the quadratic metric

$$\rho\left(f(x_1,\ldots,x_N), \hat{f}(x_1,\ldots,x_N)\right)$$
$$= \int_{\mathbf{R}^N} \left| f(x_1,\ldots,x_N) - \hat{f}(x_1,\ldots,x_N)\right|^2 dx_1\ldots dx_N,$$

where

$$\hat{f}(x_1,\ldots,x_N)$$
$$= \lim_{L_1,\ldots,L_N\to\infty} \sum_{l_1=-L_1}^{L_1} \ldots \sum_{l_N=-L_N}^{L_N} f^{L_1,\ldots,L_N}(x_1,\ldots,x_N).$$

We have various corollaries of Theorem 1 that are known from [22–29, 46–48] for 1D and 2D cases.

Corollary 1. The 1D sampling Theorem. Let us set $N = 1$ in Theorem 1. In this case, we have the sampling theorem variant formulated by Papoulis in [48]. Then when signal spectrum $f(x)$ is bounded and lies in domain $[-\Omega, \Omega]$, it can unambiguously be recovered from discrete sample values $f(k\Delta)$, $k = 0, 1, 2, \ldots$, according to the formula

$$f(x) = \sum_{k\in\mathbf{Z}} f(k\Delta) \frac{\sin(\omega_1(x - k\Delta))}{\omega_\Delta(x - k\Delta)} \quad (6)$$

if $\Delta \le \pi/\Omega$. Here, $2\omega_\Delta \ge 2\Omega$ and $\Omega \le \omega_1 \le \omega_\Delta$. When $\omega_1 = \omega_\Delta = \Omega$ series (6) becomes the known WKS relationship

$$f(x) = \sum_{k\in\mathbf{Z}} f(k\Delta) \operatorname{sinc}\left(\frac{\pi}{\Omega}(x - k\Delta)\right), \quad (7)$$

where $\operatorname{sinc}(x) = \dfrac{\sin(x)}{x}$.

The WKS Theorem states [5, 46] that signal $f(x)$ can be received at the receiver terminal when the complete infinite sequence $f(k\Delta)$ is transmitted through an ideal lowpass filter with the cutoff frequency π/Δ and the amplitude–frequency characteristic (AFC) $|K(i\omega)| = \Delta$ within the passband. However an ideal lowpass filter does not exist, and, therefore, the complete infinite sequence of sample values cannot be transmitted through it. In [5] the WKS Theorem is generalized with the use of AFs and the following series is proposed:

$$f(x) = \sum_{k\in\mathbf{Z}} f(k\Delta) F_a\left(\frac{a\pi}{\Delta}(x - k\Delta)\right),$$

where

$$F_a(t) = \prod_{j=1}^{\infty} \operatorname{sinc}\left(t/a^j\right).$$

It is shown that this choice of the interpolation kernel can enhance the accuracy of interpolation by two orders as compared to the classical series [46].

Corollary 2. The 2D sampling Theorem. In the 2D case series (2) has the following form:

$$f(x_1, x_2)$$
$$= \sum_{k_1,k_2\in\mathbf{Z}} f(k_1\Delta_1, k_2\Delta_2)\, g_B(x_1 - k_1\Delta_1, x_2 - k_2\Delta_2), \quad (8)$$

where

$$g_B(x_1, x_2) = \frac{1}{4\pi^2} \iint_B \exp\left(j\left(x_1 u_1 + x_2 u_2\right)\right) du_1 du_2.$$

Here, the behavior of function g_B is completely determined by the shape of its domain. Let us present two known examples [5] for rectangular and circular reference domains.

Example 1. As domain B let us consider the rectangle $\left[-u_{1\Delta}, u_{1\Delta}\right] \times \left[-u_{2\Delta}, u_{2\Delta}\right]$, $u_{1\Delta} = \pi/x_{1\Delta}$, $u_{2\Delta} = \pi/x_{2\Delta}$ which coincides with domain D. We obtain the 2D sampling Theorem for a rectangular frequency domain of the form

$$f\left(x_1, x_2\right) = \sum_{k_1, k_2 \in \mathbf{Z}} f\left(k_1 x_{1\Delta}, k_2 x_{2\Delta}\right) \times \operatorname{sinc}\left(u_{1\Delta}\left(x_1 - k_1 x_{1\Delta}\right)\right) \operatorname{sinc}\left(u_{2\Delta}\left(x_2 - k_2 x_{2\Delta}\right)\right). \tag{9}$$

In this case it follows from (9) that the interpolation kernel equals the product of 1D interpolation kernels

Example 2. Let domain D be square $\left[-u_{1\Delta}, u_{1\Delta}\right] \times \left[-u_{2\Delta}, u_{2\Delta}\right]$ and domain B be a circle of the radius $w_0 = u_{1\Delta} = u_{2\Delta}$. Domain A is within circle B. In this case we obtain [22–29, 46] the 2D sampling theorem for a circular reference domain of the form

$$f(x_1, x_2) = \sum_{k_1, k_2 \in \mathbf{Z}} f\left(k_1 \frac{\pi}{w_0}, k_2 \frac{\pi}{w_0}\right) \times \frac{J_1\left(\sqrt{\left(w_0 x_1 - k_1 \pi\right)^2 + \left(w_0 x_2 - k_2 \pi\right)^2}\right)}{\sqrt{\left(w_0 x_1 - k_1 \pi\right)^2 + \left(w_0 x_2 - k_2 \pi\right)^2}}, \tag{10}$$

where $J_1(x)$ is the first-order Bessel function of the first kind.

Corollary 3. The N-dimensional sampling Theorem for a rectangular frequency domain. By analogy with the 1D and 2D cases (see (7) and (9)), we obtain the N-dimensional sampling Theorem for a rectangular frequency domain

$$f\left(x_1, \dots, x_N\right) = \sum_{k_1, \dots, k_N \in \mathbf{Z}} f\left(k_1 \Delta_1, \dots, k_N \Delta_N\right) \times \prod_{j=1}^{N} \operatorname{sinc}\left(u_{j\Delta}\left(x_j - k_j x_{j\Delta}\right)\right), \tag{11}$$

where $u_{j\Delta} = \pi/x_{j\Delta}$.

B. The Generalization of the Whittaker–Kotelnikov–Shannon Sampling Theorem with the Use of Atomic Functions

We formulate the following Theorem.

Theorem 2. Let function $f(x) = f\left(x_1, \dots, x_N\right)$ have a finite spectrum ($\operatorname{supp}\hat{f}(\omega) = A \subset \mathbf{R}^N$). We choose function $\gamma(\omega) \in L_2(\mathbf{R}^N)$ such that the following conditions should be fulfilled:

(i) $\gamma(0, \dots, 0) = \dfrac{(2\pi)^N}{x_{1\Delta} \dots x_{N\Delta}}$,

(ii) $\gamma(\omega) = 0$ for $\omega \notin D$ (for points beyond domain D).

Then for any chosen discretization step $x_\Delta = \left(x_{1\Delta}, \dots, x_{2\Delta}\right) \in \mathbf{R}^N$ such that $A \subset D$ the following expansion is valid:

$$f\left(x_1, \dots, x_N\right) = \sum_{k_1, \dots, k_N \in \mathbf{Z}} \left(x_{1\Delta} \dots x_{N\Delta}\right) f\left(k_1 x_{1\Delta}, \dots, k_N x_{N\Delta}\right) \times g_B\left(x_1 - k_1 x_{1\Delta}, \dots, x_N - k_N x_{N\Delta}\right), \tag{12}$$

where

$$g_B\left(x_1, \dots, x_N\right) = \frac{1}{(2\pi)^N} \int \dots \int_B \gamma\left(\omega_1, \dots, \omega_N\right) \times \exp\left(i\left(x_1 \omega_1 + \dots + x_N \omega_N\right)\right) d\omega_1 \dots d\omega_N. \tag{13}$$

We have the domains $B \subset D$ and $A \subseteq B$.

Proof. The original signal is signal $f(x)$ with spectrum

$$F(\omega) = \begin{cases} F(\omega) & \text{when } \omega \in A, \\ 0 & \text{when } \omega \notin A. \end{cases}$$

Discretization of $f(x)$ with frequency ω_Δ yields a signal that has spectrum $\hat{F}(\omega)$ and is transmitted through ideal lowpass filter $p_B(\omega)$. At the filter output, we have processed signal $f(x)$ (2) with spectrum $F(\omega)$ (5). Next, we transmit this signal through a filter with FC $\gamma(\omega)$ to obtain the output signal

$$f_{\text{out}}\left(x_1, \dots, x_N\right) = \sum_{k_1, \dots, k_N \in \mathbf{Z}} \left(x_{1\Delta} \dots x_{N\Delta}\right) \times f\left(k_1 x_{1\Delta}, \dots, k_N x_{N\Delta}\right) g_B\left(x_1 - k_1 x_{1\Delta}, \dots, x_N - k_N x_{N\Delta}\right), \tag{14}$$

where

$$g_B\left(x_1, \dots, x_N\right) = \frac{1}{(2\pi)^N} \int \dots \int_B \gamma\left(\omega_1, \dots, \omega_N\right) \times \exp\left(i\left(x_1 \omega_1 + \dots + x_N \omega_N\right)\right) d\omega_1 \dots d\omega_N.$$

In this case we can change procedure (5) by forming the output signal spectrum $F_{\text{out}}(\omega) = \hat{F}(\omega) p_B(\omega) \gamma(\omega)$. This operation is equivalent to introducing the filter $\gamma_B(\omega) = p_B(\omega)\gamma(\omega)$ forming output signal f_{out} (14).

Setting $\gamma(\omega) = 1$ in domain B we obtain the known WKS theorem. It is known that according to the WKS theorem the discretized signal is transmitted through an ideal lowpass filter. In this case the signal is transmitted through a filter with the FC $p_B(\omega)\gamma(\omega)$, where the characteristic function of domain B is represented as

$$p_B(\omega) = \begin{cases} 1 \text{ when } \omega \in B \subset \mathbf{R}^N, \\ 0 \text{ when } \omega \notin B. \end{cases}$$

As is shown in [5, 22–29], one of the advantages of this approach (introduction of the additional filter) is in the combination of discretization and functional transformation of the original signal. Combinations of AFs and R-functions or other functions from $L^2(\mathbf{R}^N)$ satisfying the hypothesis of Theorem 1 can be chosen as function $\gamma(\omega)$. Let us obtain the Kravchenko–Kotelnikov series based on AF $fup_n(\omega)$.

C. The Generalized Kravchenko–Kotelnikov Series Based on the Fourier Transforms of Atomic Function $fup_n(\omega)$

Consider the generalized Kravchenko–Kotelnikov series where the FT of AF $fup_n(\omega)$ is used as the interpolation kernel. Atomic function $fup_n(\omega)$ [5, 16] solves the functional–differential equation

$$y'(\omega) = 2^{n+1}\sum_{k=0}^{n+2}\left(C_{n+1}^k - C_{n+1}^{k-1}\right) y\left(2^{n-1}x - \frac{2(k-1)-n}{2^{n+2}}\right).$$

With the help of the inverse FT (IFT), AF $fup_n(\omega)$ is determined as

$$fup_n(\omega) = \Im^{-1}[F(x)](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ix\omega)F(x)dx. \quad (15)$$

Here,

$$F(x) = \operatorname{sinc}^n\left(\frac{x}{2}\right)\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{x}{2^j}\right) = \Im[fup_n(\omega)](x).$$

Following [5], we present the following main properties of AF $fup_n(\omega)$:

(i) $\operatorname{supp} fup_n(\omega) = \left[-\frac{(n+2)}{2}, \frac{(n+2)}{2}\right]$,

(ii) $fup_n(\omega) \in C^\infty\left[-\frac{(n+2)}{2}, \frac{(n+2)}{2}\right]$,

(iii) $fup_n(\omega)$ is an even function,

(iv) the function is normalized as $\int_{-(n+2)/2}^{(n+2)/2} fup_n(\omega)d\omega = 1$,

(v) the derivative of function $fup_n(\omega)$ is expressed through $fup_{n-1}(\omega)$,

$$fup_n'(\omega) = fup_{n-1}\left(\omega + \frac{1}{2}\right) - fup_{n-1}\left(\omega - \frac{1}{2}\right),$$

(vi) shifts of function $fup_n(\omega)$ yield

$$\sum_{k=-\infty}^{\infty} fup_n(\omega - k) = 1.$$

Let us apply Theorem 2 to obtain an interpolation kernel of form $F(x)$. To this end, it is necessary to choose appropriately normalized function $fup_n(\omega)$ as $\gamma(\omega)$. Consider certain cases.

Case 1. Let $\gamma(\omega) = \frac{2\pi}{x_\Delta} fup_n\left(\frac{n+2}{2\Omega}\omega\right)$. Then the segment $[-\Omega, \Omega]$ is the support of this function (see property (i)). Note that Ω is the upper boundary frequency of spectrum $\hat{F}(\omega)$ (6), (7). In this case $g_\Theta(x) = g_\Omega(x)$, because

$$g_\Theta(x) = \frac{1}{x_\Delta}\int_{-\Theta}^{\Theta} fup_n\left(\frac{n+2}{2\Omega}\omega\right)\exp(ix\omega)\,d\omega$$

$$= \frac{1}{x_\Delta}\int_{-\Omega}^{\Omega} fup_n\left(\frac{n+2}{2\Omega}\omega\right)\exp(ix\omega)\,d\omega = g_\Omega(x).$$

Function $g_\Omega(x)$ can be expressed in terms of $F(x)$. Using the properties of FT (13), we obtain

$$g_\Omega(x) = \frac{1}{x_\Delta}\int_{-\Omega}^{\Omega} fup_n\left(\frac{n+2}{2\Omega}\omega\right)\exp(i\omega x)\,d\omega$$

$$= \frac{1}{x_\Delta}\Im\left[fup_n\left(\frac{n+2}{2\Omega}\omega\right)\right](-x)$$

$$= \frac{1}{x_\Delta}\Im[fup_n(\omega)]\left(\frac{2\Omega x}{n+2}\right) = \frac{1}{x_\Delta}F\left(\frac{2\Omega x}{n+2}\right).$$

Then the generalized Kravchenko–Kotelnikov series has the form

$$f(x) = \sum_{k\in\mathbf{Z}} f(kx_\Delta)\operatorname{sinc}^n\left(\frac{\Omega(x - kx_\Delta)}{(n+2)}\right) \times \prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{\Omega(x - kx_\Delta)}{2^{j-1}(n+2)}\right),$$

where $\Omega \ge \omega_\Delta$.

Case 2. If $\gamma(\omega) = \frac{2\pi}{x_\Delta} fup_n\left(\frac{n+2}{2\Omega'}\omega\right)$, $\Omega' < \Omega$ the support of this function is the segment $[-\Omega', \Omega']$. Here the signal under study cannot be recovered because a filter with FC $\gamma(\omega)$ cuts off the frequencies within the segment $[\Omega', \Omega]$.

Case 3. Assume that $\gamma(\omega) = \frac{2\pi}{x_\Delta} fup_n\left(\frac{n+2}{2\Theta}\omega\right)$, $\Omega \le \Theta \le \omega_\Delta$, and the support of this function is the segment $[-\Theta, \Theta]$. Then generalized Kravchenko–Kotelnikov series (6) takes the form

$$f(x) = \sum_{k\in\mathbf{Z}} f(kx_\Delta)\operatorname{sinc}^n\left(\frac{\Theta(x - kx_\Delta)}{(n+2)}\right) \times \prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{\Theta(x - kx_\Delta)}{2^{j-1}(n+2)}\right). \quad (16)$$

If $\Theta = \omega_\Delta = \pi/x_\Delta$, we obtain the series

$$f(x) = \sum_{k\in\mathbf{Z}} f(kx_\Delta)\,\mathrm{sinc}^n\left(\frac{\pi(x-kx_\Delta)}{x_\Delta(n+2)}\right) \times \prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(x-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right). \quad (17)$$

We retain a finite number of terms in the product from series (18) to obtain

$$f(x) = \sum_{k\in\mathbf{Z}} f(kx_\Delta)\,\varphi_{k-M}(x),$$

where

$$\varphi_{k-M}(x) = \mathrm{sinc}^n\left(\frac{\pi(x-kx_\Delta)}{x_\Delta(n+2)}\right) \times \prod_{j=1}^{M}\mathrm{sinc}\left(\frac{\pi(x-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right). \quad (18)$$

We have (18) as $M \to \infty$ and the WKS series for $n = 0$ and $M = 1$.

Case 4. When $\gamma(\omega) = \dfrac{2\pi}{x_\Delta} fup_n\left(\dfrac{n+2}{2\Omega'}\omega\right)$, $\Omega' > \omega_\Delta$ we obtain the series

$$f(x) = \sum_{k\in\mathbf{Z}} f(kx_\Delta)\,g_\Theta(x-kx_\Delta),$$

where

$$g_\Theta(x) = \int_{-\Theta}^{\Theta} fup_n\left(\frac{n+2}{2\Omega'}\omega\right)\exp(ix\omega)\,d\omega.$$

The segment $[-\Omega', \Omega']$ is the support of function $\gamma(\omega)$. Thus function $g_\Theta(x)$ does not coincide with $F(x)$. This result is due to the fact that the support of function $fup_n\left(\dfrac{n+2}{2\Omega'}\omega\right)$ is beyond $[-\Theta, \Theta]$. It follows from series (20) that $\varphi_{k-M}(x) = O\left(|x|^{-(n+M)}\right)$, $(|x| \to \infty)$. The comparison of the last expression with the relationship for the WKS series $O\left(|x|^{-1}\right)$, $(|x| \to \infty)$ enables us to draw the following conclusion: the use of (7) necessitates taking into account the number of terms of the series that is $(n + M - 1)$ times greater than the number of terms retained in the case when (18) is used. Let us exemplify this statement.

First, *we analyze series (18)*. As $|k| \to \infty$ the interpolation kernel approaches the quantity

$$\psi_n(k) = \frac{(n+2)^{n+1}\prod_{j=1}^{\infty}2^{j-1}}{|k|^{n+1}\pi^{n+1}}. \quad (19)$$

Replacing the infinite product from (19) with a finite one we obtain

$$\psi_{n,m}(k) = \frac{(n+2)^{n+1}\prod_{j=1}^{m}2^{j-1}}{|k|^{n+1}\pi^{n+1}}.$$

For the WKS series, an analogous formula has the form $\xi(k) = 1/\pi|k|$.

Let us analyze series (16). As $|k| \to \infty$ its interpolation kernel approaches the quantity

$$\psi_n(k) = \frac{(n+2)^{n+1}\prod_{j=1}^{\infty}2^{j-1}}{|k|^{n+1}\pi^{n+1}}\left(\frac{\omega_\Delta}{\Theta}\right). \quad (20)$$

We replace the infinite product from (20) with a finite one to obtain

$$\psi_{n,m}(k) = \frac{(n+2)^{n+1}\prod_{j=1}^{m}2^{j-1}}{|k|^{n+1}\pi^{n+1}}\left(\frac{\omega_\Delta}{\Theta}\right).$$

By analogy with (19), we have for series (6) $\xi'(k) = \dfrac{1}{\pi|k|}\left(\dfrac{\omega_\Delta}{\Theta}\right)$ $(\Theta \le \omega_\Delta)$. The case $\Theta = \omega_\Delta$ has been considered above. Here, (20) becomes (19).

It follows from the comparison of the results obtained for $\Theta = \omega_\Delta$ and $\Theta = \omega_\Delta/2$ [16, 22–29] that generalized Kravchenko–Kotelnikov series should be preferred to the WKS series. For example, the use of one hundred terms of the series yields $\xi(k) = 3.183 \times 10^{-3}$. The accuracy of the generalized Kravchenko–Kotelnikov series can be substantially enhanced.

D. The Physical Meaning of the Generalized Kravchenko–Kotelnikov Series

As is noted in [5, 16, 22–29, 46], according to the WKS theorem, continuous signals with a bounded spectrum (BS) can be transmitted with the help of pulse methods. This transmission system contains the following basic units:

(i) a filter with cutoff frequency Ω that limits the frequency interval of the signal spectrum,

(ii) a discretizer that makes it possible to separate from a continuous signal samples in the form of successive pulses spaced by interval x_Δ,

(iii) a communication line equipped with a pulse transmission system,

(iv) an output filter transforming the sequence of pulse samples into the continuous signal coinciding with the reference one. The filter should be linear, because its output signal is the sum of the responses (the WKS or Kravchenko–Kotelnikov kernels) produced by each individual sample pulse.

Such an ideal transmission system is difficult to realize exactly. Why? For transmitted signals to be reproduced exactly, the filter should introduce an infinitely long time delay. The complete set of the WKS or Kravchenko–Kotelnikov kernels participates in the formation of the signal value at a point situated between samples. Thus the recovery of the signal value at the point between samples necessitates receiving signal samples over an infinite time interval. The farther a sample from the observed point and the rapider the decrease of interpolation kernels and the smaller the filter's contribution to the formation of the signal at this point. Therefore it is always possible to determine the number of samples in the neighborhood of the observation point that provide for the signal recovery with the necessary accuracy.

E. The Generalized Kravchenko–Kotelnikov Theorem for the Case of a Time Shift of a BS Signal

Let us analyze BS signal $f(x+\alpha)$ with spectrum width 2Ω. In the presence of a time shift the width of the signal spectrum is retained according to the properties of the FT. Then, by analogy with (18), we obtain the series

$$f(x+\alpha) = \sum_{k\in\mathbf{Z}} f(kx_\Delta + \alpha)\,\mathrm{sinc}^n\left(\frac{\pi(x-kx_\Delta)}{x_\Delta(n+2)}\right) \times \prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(x-kx_\Delta)}{x_\Delta\times 2^{j-1}(n+2)}\right). \tag{21}$$

Let us exemplify the application of new series (21). Let $f(x) = \mathrm{sinc}(\Omega' x)$, $\Omega \le \Omega' \le \omega_\Delta$. Then it follows from (21) that

$$\mathrm{sinc}(\Omega'(x+\alpha)) = \sum_{k\in\mathbf{Z}}\mathrm{sinc}(\Omega'(kx_\Delta+\alpha)) \times \mathrm{sinc}^n\left(\frac{\pi(x-kx_\Delta)}{x_\Delta(n+2)}\right)\prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(x-kx_\Delta)}{x_\Delta\times 2^{j-1}(n+2)}\right).$$

This relationship is valid for any x and α. When $x = -\alpha$, $\Omega' = \dfrac{\pi}{x_\Delta(n+2)} = \dfrac{\omega_\Delta}{n+2}$, and $\Omega \le \Omega' \le \omega_\Delta$ we obtain conditions imposed on the choice of n ($n \le (\omega_\Delta/\Omega) - 2$). Let us introduce the variable $\xi = \Omega'\alpha$. Then

$$\sum_{k\in\mathbf{Z}}\mathrm{sinc}^{n+1}\left(\xi\left(1+k\frac{x_\Delta}{\alpha}\right)\right)\prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\xi}{2^{j-1}}\left(1+k\frac{x_\Delta}{\alpha}\right)\right) = 1.$$

Let us estimate the errors caused by the truncation of the generalized Kravchenko–Kotelnikov series. For the sake of obviousness, series (21) can be represented as follows:

$$f(x+\alpha) = \sum_{k\in\mathbf{Z}} f(kx_\Delta + x)\,\mathrm{sinc}^n\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta(n+2)}\right) \times \prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right).$$

When a finite number of terms are retained in this expansion the error is

$$e_K(x+\alpha) = f(x+\alpha) - \sum_{k=-K}^{K} f(kx_\Delta + x) \times \mathrm{sinc}^n\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta(n+2)}\right)\prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right).$$

This error can be regarded as the output signal of a system that has the FC

$$H(\omega) = \exp(i\omega x) - \sum_{k=-K}^{K}\exp(ikx_\Delta\omega) \times \mathrm{sinc}^n\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta(n+2)}\right)\prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right),$$

and input signal $f(x)$. We use the following expression from [16]:

$$|f(x)|^2 \le \frac{E}{2\pi}\int_{-\Omega}^{\Omega}|H(\omega)|^2\,d\omega.$$

Here, $E = \int_{\mathbf{R}} f(x)^2 dx$ is the energy of signal $f(x)$. Then the inequality

$$|e_K(x+\alpha)|^2 \le \frac{E}{2\pi}\int_{-\Omega}^{\Omega}\left|\exp(i\omega x) - \sum_{k=-K}^{K}\exp(ikx_\Delta\omega) \times \mathrm{sinc}^n\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta(n+2)}\right)\prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right)\right|^2 d\omega$$

is valid. This result can be explained as follows. If the exponential function $\exp(i\omega x)$ can be expanded in the Fourier series on interval $[-\Omega, \Omega]$ the coefficients of the expansion have the form

$$a_n = \mathrm{sinc}^n\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta(n+2)}\right)\prod_{j=1}^{\infty}\mathrm{sinc}\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right).$$

We apply Parseval's formula to obtain

$$|e_K(x+\alpha)|^2 \le \frac{E\Omega}{\pi}\sum_{|k|>K}\mathrm{sinc}^{2n}\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta(n+2)}\right) \times \prod_{j=1}^{\infty}\mathrm{sinc}^2\left(\frac{\pi(\alpha-kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right). \tag{22}$$

A similar formula for the WKS series has the form

$$\left|e_K(x+\alpha)\right|^2 \le \frac{E\Omega}{\pi}\sum_{|k|>K}\operatorname{sinc}^2\left(\frac{\pi(\alpha - kx_\Delta)}{x_\Delta}\right). \quad (23)$$

As has been shown above, since the generalized Kravchenko–Kotelnikov series converges rapider than the WKS series, truncation error in (22) is substantially smaller than that in (23).

Consider another example. Let us denote

$$\Psi_k(\alpha) = \sum_{|k|>K}\operatorname{sinc}^2\left(\frac{\pi(\alpha - kx_\Delta)}{x_\Delta}\right),$$

$$\Psi_{kk}(\alpha) = \sum_{|k|>K}\operatorname{sinc}^{2n}\left(\frac{\pi(\alpha - kx_\Delta)}{x_\Delta(n+2)}\right) \times \prod_{j=1}^{\infty}\operatorname{sinc}^2\left(\frac{\pi(\alpha - kx_\Delta)}{x_\Delta 2^{j-1}(n+2)}\right).$$

We specify a certain value of x_Δ (e.g., 0.2 s), set $K = 10$, and plot dependences $\Psi_k(\alpha)$ and $\Psi_{kk}(\alpha)$. For comparatively small values of α (–1.6 to 1.6), $\Psi_{kk}(\alpha)$ is zero (even at the points that are not sampling ones) and function $\Psi_k(\alpha)$ exhibits oscillations. Let us investigate the generalized Kravchenko–Kotelnikov Theorem in the frequency domain.

F. The Generalized Kravchenko–Kotelnikov Theorem in the Frequency Domain

In practice, signal spectrum $\hat{F}(\omega)$ very often has to be represented by discrete samples, for example, when a receiver–transmitter contains digital filters. For spectrum $\hat{F}(\omega)$ we have a series similar to (17) and (18). The following Theorem is valid.

Theorem 3. Let signal $f(x) \in L^2(\mathbf{R})$ have a finite duration

$$f(x) = \begin{cases} f(x), & |x| \le \rho, \\ 0, & |x| > \rho. \end{cases}$$

Then, its spectrum $\hat{F}(\omega)$ can be represented by the series

$$\hat{F}(\omega) = \sum_{k\in\mathbf{Z}}\hat{F}(k\Delta\omega_\Delta)\operatorname{sinc}^n\left(\frac{\rho(x - k\Delta\omega_\Delta)}{(n+2)}\right) \times \prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{\rho(x - k\Delta\omega_\Delta)}{2^{j-1}(n+2)}\right),$$

where $\Delta\omega_\Delta$ is the frequency sampling step satisfying the condition $\Delta\omega_\Delta \le \pi/\rho$.

Theorem 3 can be proved by analogy with Theorem 2.

G. The Gibbs Phenomenon and Kravchenko Weighting Functions Based on Atomic Function $fup_n(k)$

Let us investigate physical phenomena [16] related with the Gibbs effect and demonstrate methods that can be applied to remove it. In practice, this technique is realized by introducing WFs (windows) [16, 22–29]. Let spectrum $F(\omega)$ of original BS signal have discontinuities of the first kind at the boundary points $|\omega| = \Omega$. The periodic extension of this spectrum with the frequency $2\omega_\Delta$ yields spectrum $\hat{F}(\omega)$

$$\hat{F}(\omega) = \sum_{k\in\mathbf{Z}} F(\omega - k\omega_\Delta), \quad \omega_\Delta \ge \Omega.$$

As $k \to \infty$, the series does not converge to $F(\omega)$ at the discontinuity points, a circumstance that corresponds to the known physical Gibbs phenomenon which is due to the fact that the Fourier series minimizes the root-mean-square error(RMSE).

Therefore the shape of the function that is expanded in the Fourier series cannot be restored. When $\omega_\Delta = \Omega$, the Gibbs effect does not occur, because the periodic extension is performed by joining discontinuities. Thus, when $\omega_\Delta > \Omega$, the Gibbs phenomenon distorting the real spectrum is observed. We consider certain issues related with this physical phenomenon by analogy with studies [16, 22–29, 46].

Assume that only a segment of function $f(x)$

$$f_W(x) = W(x)f(x) \quad (24)$$

is known. Here, function $W(x) \ge 0$ is called a WF (window) of function $f(x)$. The spectrum of such cutoff function $f_W(x)$ is equal to the spectral convolution

$$F_W(\omega) = \int_{\mathbf{R}} F(\omega - \omega')K_W(\omega')d\omega', \quad (25)$$

where $K_W(\omega)$ is the spectrum of WF $W(x)$. In the case of periodic function $\hat{F}(\omega)$ represented by (24), we have for the generalized Kravchenko–Kotelnikov series

$$\hat{F}(\omega) = x_\Delta\sum_{k\in\mathbf{Z}} f(kx_\Delta)\gamma(\omega)\exp(-ikx_\Delta\omega). \quad (26)$$

Series (26) is the spectrum of signal (12) ($N = 1$). As the original signal we can use (16)–(18) with corresponding functions $\gamma(\omega)$. Assume that segment $f(kx_\Delta)$ is known and WF W_k that is nonzero within interval $(-K, K)$ is introduced. Then

$$\hat{F}_K(\omega) = x_\Delta\sum_{k=-K}^{K} W_k f(kx_\Delta)\gamma(\omega)\exp(-ikx_\Delta\omega). \quad (27)$$

Expression (27) establishes the relationship between $(2K+1)$ sample values of function $f(x)$ at the points kx_Δ and its spectrum $\hat{F}_K(\omega)$ for differently weighted sample values. Since (27) is a discrete FT of the product of two functions, $f(kx_\Delta)\gamma(\omega)$ and W_k, $\hat{F}_K(\omega)$ can be represented in the form of the convolution of their spectra

$$\hat{F}_K(\omega) = \int_{-\omega_\Delta}^{\omega_\Delta} \hat{F}(\omega')\, K_{W_K}(\omega - \omega')\, d\omega', \tag{28}$$

where K_{W_K} is the spectrum of the WF for $(2K+1)$ samples.

According to [5–9, 16], WFs (windows) are widely used in practice. Consider two of these

(i) a rectangular window

$$W_k = \begin{cases} 1, & -K \le k \le K, \\ 0, & |k| > K, \end{cases}$$
$$\to K_{W_k}(\omega) = \frac{x_\Delta}{2\pi} \frac{\sin\left(K \frac{x_\Delta \omega}{2}\right)}{\sin(x_\Delta \omega/2)} \tag{29}$$

and (ii) a triangular (Feyer) window

$$W_k = \begin{cases} 1 - \frac{|k|}{K}, & -K \le k \le K, \\ 0, & |k| > K, \end{cases}$$
$$\to K_{WW_k}(\omega) = \frac{x_\Delta}{2\pi} \frac{\sin^2\left(K \frac{x_\Delta \omega}{2}\right)}{\sin^2(x_\Delta \omega/2)}. \tag{30}$$

We introduce the Kravchenko weighting function (window) based on AF $fup_n(k)$. We use property (ii) for the support of function $fup_n(k)$ to obtain the following WF:

$$W_{fup_k} = \begin{cases} fup_n(k), & -K \le k \le K, \\ 0, & |k| > K, \end{cases} \quad K = \frac{n+2}{2},$$
$$K_{W_{fup}}(\omega) = \mathrm{sinc}^n\left(\frac{\omega}{2}\right) \prod_{j=1}^{\infty} \mathrm{sinc}\left(\frac{\omega}{2^j}\right). \tag{31}$$

Thus, when it is necessary to apply finite number K we can choose $n = 2K - 2$ for kernels of (31). It follows from (28) that WF (window) (29) corresponds to a finite sum in the WKS Theorem which in the case when spectral analysis is performed for a discontinuous spectrum, leads to the Gibbs effect. According to [16, 22–29], the Gibbs phenomenon is removed when a triangular window is used, because the WF spectrum is positive. This physical phenomenon occurs when spectral analysis is performed for random processes ($F(\omega) \ge 0$). The application of a rectangular window can lead to its negative spectrum. Then, this phenomenon is absent as in the case of WF (31).

H. The generalized Kravchenko–Kotelnikov Theorem for a Bounded–Spectrum Bandpass Signal

Let the narrowband BS signal

$$f(x) = g(x)\cos(\omega_0 x + \Theta(x)) \tag{32}$$

be specified. Here, $\hat{g}(\omega) = 0$ for $|\omega| > \omega_f$ and $2\omega_f$ is the boundary frequency.

Signal (31) is associated with the analytical signal

$$F(x) = f(x) + i\gamma(x) = G(x)\exp(i\omega_0 x),$$

where $\gamma(x) = \frac{1}{\pi}\int_{\mathbf{R}} \frac{f(\xi)}{x - \xi} d\xi$ is a function Hilbert-conjugate to $f(x)$. The complex envelope of signal $f(x)$ has the form $G(x) = g(x)\exp(i\Theta(x))$. With the help of quadrature components the signal can be represented as

$$f(x) = f_1(x)\cos(\omega_0 x) - f_2(x)\sin(\omega_0 x),$$

where $f_1(x) = g(x)\cos(\Theta(x))$ and $f_2(x) = g(x)\sin(\Theta(x))$. We represent envelope $G(x)$ as $G(x) = f_1(x) + if_2(x)$. Since $f(x)$ is the real part of $F(x)$ we have

$$f(x) = \frac{1}{2}(F(x) + F^*(x))$$
$$= \frac{1}{2}(G(x)\exp(i\omega_0 x) + G^*(x)\exp(-i\omega_0 x)).$$

The spectrum of the signal is

$$S_f(\omega) = \frac{1}{2}\left(S_G(\omega - \omega_0) + S_G^*(-\omega - \omega_0)\right).$$

Here, $\frac{1}{2}S_G(\omega - \omega_0)$ corresponds to the interval of spectrum $S_f(\omega)$ lying in the region of positive frequencies, $\frac{1}{2}S_G(-\omega - \omega_0)$ corresponds to the interval lying in the region of negative frequencies, and $S_G(\omega)$ is the spectrum of BS function $G(x)$, i.e., $S_G(\omega) = 0$ when $|\omega| > \Omega'$.

With the help of the generalized Kravchenko–Kotelnikov Theorem, function $G(x)$ can be represented in the form

$$G(x) = \sum_{k \in \mathbf{Z}} G(kx_\Delta)\,\mathrm{sinc}^n\left(\frac{\Omega'(x - kx_\Delta)}{(n+2)}\right) \times \prod_{j=1}^{\infty} \mathrm{sinc}\left(\frac{\Omega'(x - kx_\Delta)}{2^{j-1}(n+2)}\right), \tag{33}$$

where $\Omega' = \omega_f/2$ and x_Δ is the dicretization step. Thus, we have for high-frequency signal $f(x)$

$$f(x) = \mathrm{Re}\left[G(x)\exp(i\omega_0 x)\right]$$
$$= \mathrm{Re}\left[\sum_{k \in \mathbf{Z}} G(kx_\Delta)\exp(i\omega_0 x)\,\mathrm{sinc}^n\left(\frac{\Omega'(x - kx_\Delta)}{(n+2)}\right) \times \prod_{j=1}^{\infty} \mathrm{sinc}\left(\frac{\Omega'(x - kx_\Delta)}{2^{j-1}(n+2)}\right)\right] = \sum_{k \in \mathbf{Z}} g(kx_\Delta)\cos(\omega_0 x + \Theta(kx_\Delta))$$
$$\times\ \mathrm{sinc}^n\left(\frac{\Omega'(x - kx_\Delta)}{(n+2)}\right) \prod_{j=1}^{\infty} \mathrm{sinc}\left(\frac{\Omega'(x - kx_\Delta)}{2^{j-1}(n+2)}\right).$$

The amplitude and phase of a BS signal should be discretized with frequency $\omega_\Delta \geq \Delta\omega_f$ to eliminate the superposition effect.

I. On the Further Application of the Generalized Kravchenko–Kotelnikov Theorem in Problems of the Information and Communication Theory

The generalized Kravchenko–Kotelnikov sampling theorem can be extended to the theory of random processes and signals [16] on the basis of the following Theorem.

Theorem 4. Let $x(t)$, $t \in \mathbf{R}$ be a stochastic process that is stationary in the wide sense and has a spectral density vanishing beyond the interval $[-2\pi\Omega, 2\pi\Omega]$. Then, for arbitrary $t \in \mathbf{R}$, $x(t)$ has the form

$$x(t) = \lim_{K\to\infty} \sum_{k=-K}^{K} x\left(\frac{k}{2\Omega}\right) \operatorname{sinc}^n\left(\frac{\pi(2\Omega t - k)}{(n+2)}\right) \times \prod_{j=1}^{\infty} \operatorname{sinc}\left(\frac{\pi(2\Omega t - k)}{2^{j-1}(n+2)}\right). \quad (34)$$

Proof. The proof is based on the use of the generalized Kravchenko–Kotelnikov sampling theorem for the covariation function of process $x(t)$. Variants of its application to the Shannon information theory are possible. For example, let $n(t) = \sum_{l\in\mathbf{Z}} n_l \xi_l(t)$ be an orthogonal decomposition of a noise on interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$ where

$$\frac{1}{T}\int_{-T/2}^{T/2} \xi_l(t)\xi_m(t)dt = \begin{cases} 1 \text{ when } l = m, \\ 0 \text{ when } l \neq m. \end{cases}$$

Quantities n_l are characterized by Gausian distributions with the zero means and the variances $\sigma_\xi^2 = N_0/2T$. Assume that useful signal $x(t)$ with a bounded spectrum in band δF is transmitted over a channel with noise $n(t)$. This signal is summed with the noise and forms the signal–noise mixture $y(t) = x(t) + n(t)$. Signals $x(t)$, $y(t)$, and $n(t)$ are determined by samples in a $2T\delta F$-dimensional space according to the generalized Kravchenko–Kotelnikov theorem. Let $p(s)$ be the probability density function (PDF) of samples s_l. Then, when $\int_{s_l\in S} |\hat{S}|^2 p(s)dS \leq P_s$, the capability is $C = \delta F \log\left(1 + \frac{P_s}{N_0\delta F}\right)$ where δF is the frequency bandwidth, $N_0\delta F$ is the mean noise power in this band and P_s is the useful signal power.

3. ATOMIC FUNCTIONS IN THE THEORY OF PROBABILITY AND RANDOM PROCESSES

A. The Probabilistic Properties of Parent Atomic Function $up(x)$

Consider the basic notions related with AF $up(x)$ [5] and its probabilistic properties [33–40]. We ask the following question: If there is a sequence of heads-and-tails discrete random quantities (RFs), how a uniform distribution can be obtained? Let $\{\xi_k\}_{k=1}^N$ be a sequence of independent discrete RQs taking the values –1 and 1 with the probability 0.5. Random quantity ξ^N can be represented in the form

$$\xi^N = \sum_{k=1}^{N} \xi_k \times 2^{-k}, \quad \xi_k \in \{-1, 1\}. \quad (35)$$

Let $\{\xi_k\}_{k=1}^\infty$ be a sequence of independent RQs uniformly distributed on [–1, 1]. We ask the following question: What PDF does the RQ

$$\xi = \sum_{k=1}^{\infty} \xi_k \times 2^{-k} \quad (36)$$

have? To answer this question, we use the property of the PDF of a sum of two RQs [30–40]. Let RQ X (Y) have PDF $p_X(x)$ ($p_Y(y)$). Then, the RQ $Z = X + Y$ has the PDF

$$p_Z(z) = \int_{\mathbf{R}} p_X(z-\lambda)\, p_Y(\lambda)\, d\lambda. \quad (37)$$

We apply formula (37) to calculate the PDF of the RQ specified by equality (36).

The convolution of functions $p(\zeta_1)$ and $p(\zeta_2)$ yields PDF $p(\zeta_{12})$ having a trapezoidal shape. Next the convolution of functions $p(\zeta_{12})$ and $p(\zeta_3)$ yields a smoothed finite function with the support [–0.875, 0.875]. Repeating the convolution procedure an infinite number of times, we obtain a smooth function. We denote this PDF $up(\xi)$. Let us show that $up(\xi)$ solves the functional–differential equation

$$up'(\xi) = 2up(2\xi + 1) - 2up(2\xi - 1). \quad (38)$$

Function $up(\xi)$, introduced above as the convolution of an infinite number of rectangular pulses, should satisfy the following theorem.

Theorem 5. Probability density function $up(\xi)$ of RQ ξ solves Eq. (38) and is analytically expressed as

$$up(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}} \prod_{j=1}^{\infty} \operatorname{sinc}\left(\frac{t}{2^j}\right) \exp(it\xi)\, dt. \quad (39)$$

Proof. Since $up(\xi)$ is the convolution of an infinite number of rectangular pulses, characteristic function (CF) $F(t)$ of RQ ξ can be represented in the form

$$F(t) = \prod_{j=1}^{\infty} \operatorname{sinc}\left(\frac{t}{2^j}\right). \quad (40)$$

The PDF of RQ ξ is found with the help of the IFT of CF (40). Thus, the PDF of RQ ξ has form (39). Let us show that (39) solves (38). To this end we differentiate (39) to obtain

$$up'(\xi) = \frac{1}{2\pi} \int_{\mathbf{R}} (it) F(t) \exp(it\xi)\, dt. \quad (41)$$

Let us prove that the function $2up(2\xi+1) - 2up(2\xi-1)$ equals (41). Actually

$$\begin{aligned} &2up(2\xi+1) - 2up(2\xi-1) \\ &= 2\left(\frac{1}{2\pi}\int_{\mathbf{R}}\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{2^j}\right)\exp(it(2\xi+1))\,dt - \frac{1}{2\pi}\int_{\mathbf{R}}\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{2^j}\right)\exp(it(2\xi-1))\,dt\right) \\ &= 2\left(\frac{1}{2\pi}\int_{\mathbf{R}}\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{2^j}\right)(\exp(it(2\xi+1)) - \exp(it(2\xi-1)))dt\right) = 2\left(\frac{1}{2\pi}\int_{\mathbf{R}}\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{2^j}\right) \times \exp(2it\xi)(\exp(it)-\exp(-it))\,dt\right) \\ &= 2\left(\frac{1}{2\pi}\int_{\mathbf{R}}(2it)\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{2^j}\right) \times \exp(2it\xi)\frac{(\exp(it)-\exp(-it))}{2it}\,dt\right) \\ &= 2\left(\frac{1}{2\pi}\int_{\mathbf{R}}(2it)\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{2^j}\right)\exp(2it\xi)\operatorname{sinc}(t)\,dt\right). \end{aligned}$$

In the last expression we replace the variable according to the formula $\tau = 2t$. Then

$$\begin{aligned} &2up(2\xi+1) - 2up(2\xi-1) \\ &= \frac{1}{2\pi}\int_{\mathbf{R}}(i\tau)F\left(\frac{\tau}{2}\right)\exp(i\tau\xi)\operatorname{sinc}\left(\frac{\tau}{2}\right)d\tau. \end{aligned}$$

Using in the last integral the property $F(\tau) = F\left(\frac{\tau}{2}\right)\operatorname{sinc}\left(\frac{\tau}{2}\right)$, we obtain the relationship

$$2up(2\xi+1) - 2up(2\xi-1) = \frac{1}{2\pi}\int(i\tau)F(\tau)\exp(i\tau\xi)d\tau$$

which coincides with (10). The Theorem is proven.

We put the question: What PDF does the RQ

$$\xi = \sum_{k=1}^{\infty} \xi_k a^{-k} \quad (42)$$

have? In (42) $\{\xi_k\}\big|_{k=1}^{\infty}$ is a sequence of RQs uniformly distributed on $[-1, 1]$ and $a > 1$. We answer this question proving the following limit Theorem.

Theorem 6. Let a sequence of independent RQs $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ uniformly distributed on $[-1, 1]$ be specified. Then, for any constant $a > 1$ the PDF of the weighted sum $\xi = \frac{\xi_1}{a} + \frac{\xi_2}{a^2} + \ldots + \frac{\xi_n}{a^n} + \ldots$ has the form

$$h_a(\xi) = \frac{1}{2\pi}\int_{\mathbf{R}}\prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{a^j}\right)\exp(it\xi)\,dt \quad (43)$$

with the support $\left[-\frac{1}{a-1}, \frac{1}{a-1}\right]$.

Proof. It follows from the construction of $up(\xi)$ and Theorem 5 that CF $F_a(t)$ of RQ (43) is

$$F_a(t) = \prod_{j=1}^{\infty}\operatorname{sinc}\left(\frac{t}{a^j}\right). \quad (44)$$

It follows from (44) that the PDF of RQ (42) has form (43). Let us determine the support of PDF $h_a(\xi)$. We represent (42) in the form

$$\xi(a) = \sum_{k=1}^{\infty}\xi_k a^{-k}. \quad (45)$$

Obviously $\operatorname{supp}[h_a(\xi)] = [-l, l]$ where

$$l = \sum\nolimits_{j=1}^{\infty} a^{-j} = \frac{1}{a-1}.$$

The theorem is proven.

By analogy with Theorem 5 we can show that $h_a(\xi)$ solves the functional–differential equation

$$h_a'(\xi) = -\frac{a^2}{2}\{h_a(a\xi-1) - h_a(a\xi+1)\}. \quad (46)$$

Functions $up(\xi)$ and $h_a(\xi)$ belong to the class of AFs.

Definition. Atomic functions are finite solutions to functional–differential equations of the form

$$Ly(x) = \lambda\sum_{k=1}^{M} c_k y(ax - b_k), \quad (47)$$

where $a > 1$ and L is a linear ordinary differential operator with constant coefficients. Equations (45) and (46) are special cases of (47). Operator (47) is comprehensively investigated in [5–9, 17].

B. The Kravchenko–Rvachev Atomic Distribution and Its Moments

Let us define Kravchenko–Rvachev PDF $p_X(x)$ of RQ X as

$$p_X(x) = \frac{1}{b} h_a\left(\frac{x-m}{b}\right). \quad (48)$$

It is known from [5] that

$$\int_{\mathbf{R}} h_a(x)\,dx = \int_{-(a-1)^{-1}}^{(a-1)^{-1}} h_a(x)\,dx = 1. \quad (49)$$

Taking into account (49) we see that $\int_{\mathbf{R}} p_X(x)\,dx = 1$. Let us find the mean of RQ X having PDF $p_a(x)$. Actually

$$m_X = \int_{\mathbf{R}} x p_X(x)\,dx = \int_{-(a-1)^{-1}}^{(a-1)^{-1}} x \frac{1}{b} h_a\left(\frac{x-m}{b}\right) dx. \quad (50)$$

Let us change variables in (50) according to the formulas $\tau = (x-m)/b$ and $x = b\tau + m$. Then $m_X = m$. We determine the moments of the RQ with PDF (48). To this end we find the moments of PDF $h_a(x)$. It follows from (44) that CF $F_a(t)$ of RQ X exhibits the property

$$F_a(t) = \operatorname{sinc}\left(\frac{t}{a}\right) F_a\left(\frac{t}{a}\right). \quad (51)$$

Let $F_a(t) = \sum_{k=0}^{\infty} c_k(a) t^k$. It follows from (45) that all of the odd moments are zeros, $c_{2k+1}(a) = 0$. Thus,

$$F_a(t) = \sum_{k=0}^{\infty} c_{2k}(a) t^{2k}.$$

Since

$$\operatorname{sinc}\left(\frac{t}{a}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k} a^{-2k}}{(2k+1)!}$$

(51) implies that

$$c_{2k}(a) = \sum_{j=0}^{k} \frac{c_{2k-2j}(-1)^j a^{-2j}}{a^{2k-2j}(2j+1)!}.$$

Thus,

$$c_{2k}(a) = \frac{1}{a^{2k}-1} \sum_{i=0}^{k-1} \frac{c_{2i}(a)(-1)^{k-i}}{(2k-2i+1)!}.$$

Since

$$c_{2k}(a) = \frac{F_a^{(2k)}(0)}{(2k)!} \text{ and } F_a^{(2k)}(0) = (-1)^k \int_{\mathbf{R}} x^{2k} h_a(x)\,dx,$$

moments $m_{2k}(a)$ of function $h_a(x)$ are represented in the form

$$m_{2k}(a) = \frac{(2k)!}{a^{2k}-1} \sum_{i=0}^{k-1} \frac{c_{2i}(a)(-1)^{k-i}}{(2k-2i+1)!}.$$

We present the first ten moments for the special case when $a = 2$ and $h_a(x) = up(x)$

$$m_0(2) = 1, \quad m_2(2) = \frac{1}{3^2} = 0.111111\ldots,$$

$$m_4(2) = \frac{19}{3^3 \times 5^2} = 0.028148\ldots,$$

$$m_6(2) = \frac{11 \times 53}{3^5 5 \times 7^2} = 0.0097925590\ldots,$$

$$m_8(2) = \frac{59 \times 2251}{3^7 \times 5^3 \times 7 \times 17} = 0.0040824582,$$

$$m_{10}(2) = \frac{46840699}{3^7 5^2 \times 7 \times 11^2 \times 17 \times 31} = 0.0019192898.$$

Here, the variance of RQ X with PDF $up(x)$ is 1/9. When the PDF of RQ X has form (48), its variance is $\sigma^2 = b^2/9$ and the root-mean-square deviation (RMSD) is $\sigma = b/3$. We are mainly interested in PDF $up(x)$ rather than $h_a(x)$. All the reasoning on $up(x)$ can be extended to other AFs [5].

C. The Asymmetry, Excess, and Entropy of a Random Quantity Having the Kravchenko–Rvachev Probability Density Function

Let us find the basic numerical characteristics (except moments) of a RQ having the Kravchenko–Rvachev PDF. According to the definition [30–40] the ratio of third central moment m_3 to the cubed RMSD $A = m_3/\sigma^3$ is called asymmetry A of RQ X. In our case $A = 0$. The ratio of fourth central moment m_4 to the squared variance minus the number 3 $E = m_4/\sigma^4 - 3$ is called excess E of RQ X. Let us find the numerical value of the excess for an RQ with density $up(x)$: $E = -3^2 \times 2/5^2 = -0.72$. The number $H = H(X)$ is called entropy $H = H(X) = -\int_{\mathbf{R}} p_X(x) \log_2(p_X(x))\,dx$. In our case the entropy is $H = 0.41328$ for the logarithm base 2.

The Kravchenko–Rvachev distribution function. Let us find the distribution function corresponding to (48) under the condition that $a = 2$. By the definition [30–32] we have $F_X(x) = \int_{-\infty}^{x} p_X(\omega)\,d\omega$, where $p_X(x) = [dF_X(x)]/dx$. Actually

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{b} up\left(\frac{\omega}{b}\right) d\omega = \int_{-b}^{x} \frac{1}{b} up\left(\frac{\omega}{b}\right) d\omega = \int_{-1}^{x/b} up(\xi)\,d\xi. \quad (52)$$

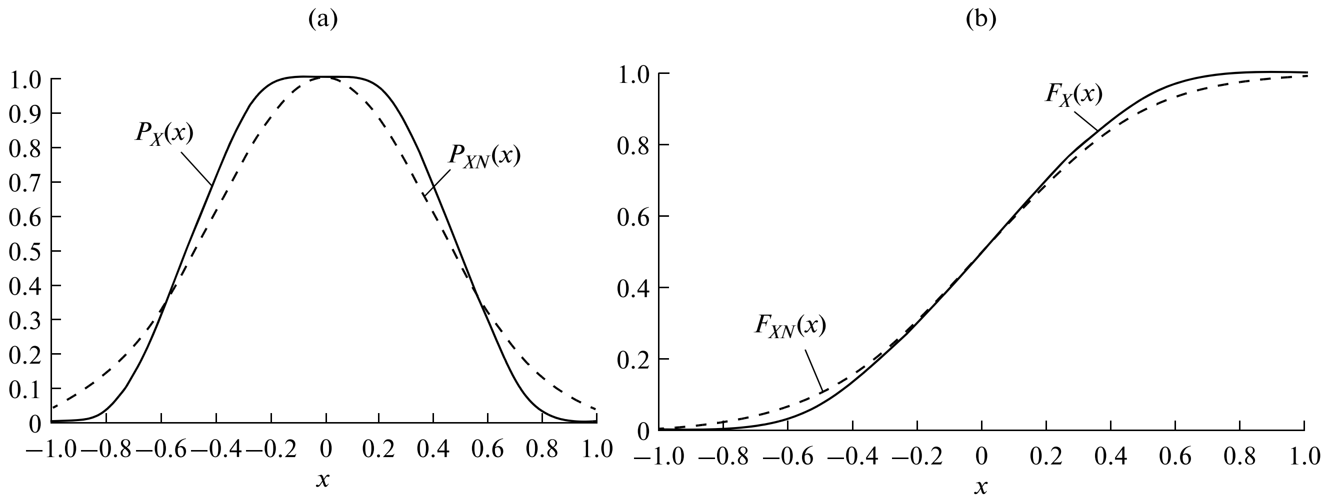

Fig. 2. Behavior of the (a) probability density and (b) distribution function: the (solid line) Kravchenko–Rvachev and (dashed line) Gaussian dependences.

We can represent (52) in the form

$$F_X(x) = 1 - Q_X(x/b), \tag{53}$$

where

$$Q_\theta(x) = \int_x^1 up(\xi)\,d\xi. \tag{54}$$

Distribution (52) is below referred to as the Kravchenko–Rvachev distribution. Expressions (53) and (54) have a form similar to the Gaussian (normal) distribution such that

$$p_{XN}(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-x^2/2\sigma^2\right), \tag{55}$$

$$F_{XN}(x) = 1 - Q_{XN}(x/\sigma), \tag{56}$$

where

$$Q_{XN}(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\xi^2/2\right)d\xi. \tag{57}$$

Figure 2 displays PDFs (48) and (55) and distribution functions (53) and (57). We set $\sigma = 1/\sqrt{2\pi}$ in (55) to obtain $p_{XN}(0) = 1$.

The Chebyshev inequality. Consider the Chebyshev inequality. Let θ be an RQ with mean m, variance σ_θ^2, and PDF $p_\theta(x)$. Then,

$$P_\theta\left[|x - m| \geq c\sigma_\theta\right] \leq \frac{1}{c^2}, \tag{58}$$

where $P_\theta[\alpha \geq \beta]$ is the probability of the fact that $\alpha \geq \beta$.

When $c = 2.3$, the following probabilistic statements can be formulated on the basis of (58): $P_\theta\left[|x - m| \geq 2\sigma_\theta\right] \leq 0.250$ and $P_\theta\left[|x - m| \geq 3\sigma_\theta\right] \leq 0.111$. Let us compare these results for the Gaussian [31, 32] and Kravchenko–Rvachev distributions. We have

$$P_\theta\left[|x - m| \geq 2\sigma_\theta\right] < 0.050, \quad P_\theta\left[|x - m| \geq 3\sigma_\theta\right] < 0.003$$

for the Gaussian distribution and

$$P_\theta\left[|x - m| \geq 2\sigma_\theta\right] < 0.030, \quad P_\theta\left[|x - m| \geq 3\sigma_\theta\right] = 0$$

for the Kravchenko–Rvachev distribution. The last expression is valid, because the domain of the Kravchenko–Rvachev PDF is bounded by value b and $\sigma_\theta = \sqrt{\sigma_\theta^2} = b/3$.

D. N-Dimensional Probability Density Function, Statistical Independence, and Mixed Expressions for Probabilities

Let us define the PDF [31, 32] of random N-dimensional vector $x = (x_1, x_2, \ldots, x_N) \in \mathbf{R}^N$ for any domain $O \subset \mathbf{R}^N$ as follows:

$$P\left[\{\omega : x(\omega) \in O\}\right] = \int_O p_x(\xi)\,d\xi. \tag{59}$$

If we assume that O is a domain where $x \leq \mu$, $\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbf{R}^N$, (59) yields

$$F_x(\mu) = P\left[\{\omega : x(\omega) \leq \mu\}\right] = \int_{-\infty}^{\mu_N} \ldots \int_{-\infty}^{\mu_2}\int_{-\infty}^{\mu_1} p_x(\xi)\,d\xi_1 d\xi_2 \ldots d\xi_N. \tag{60}$$

If $F_x(\mu)$ is a function continuous and differentiable at point μ, differentiation of (60) with respect to the upper integration limits yields

$$p_x(\mu) = \frac{\partial^N}{\partial\mu_1\partial\mu_2\ldots\partial\mu_N} F_x(\mu). \tag{61}$$

It is known [30–32] that RQs $x_1, x_2, \ldots, x_n$ are statistically independent when joint density function p_x of these quantities equals the product

$$\prod_{i=1}^{N} p_{x_i}. \tag{62}$$

If each RQ x_i has a PDF of form (48) then (62) is represented as

$$p_x(\mu_1, \ldots, \mu_N) = \prod_{i=1}^{N} \frac{1}{b_i} up\left(\frac{\mu_i - m_i}{b_i}\right). \tag{63}$$

In the particular case when $N = 2$, (63) can be expressed as

$$p_x(\mu_1, \mu_2) = \frac{1}{b_1 b_2} up\left(\frac{\mu_1 - m_1}{b_1}\right) up\left(\frac{\mu_2 - m_2}{b_2}\right). \tag{64}$$

E. Cumulant Analysis of the Kravchenko–Rvachev Atomic Distribution

Characteristic function $F_\theta(\xi)$ of RQ θ can be represented as $F_\theta(\xi) = \exp\{B_\theta(\xi)\}$, where the equality $B_\theta(0) = 0$ is fulfilled. Let us expand function $B_\theta(\xi)$ in the power series

$$B_\theta(\xi) = \ln\{F_\theta(\xi)\} = \sum_{k=1}^{\infty} \frac{\chi_k}{k!}(i\xi)^k. \tag{65}$$

The coefficients of series (65) are such that

$$\chi_k = (i)^{-k} B_\theta^{(k)}(0) = (i)^{-k} \left[\frac{d^k \ln(F_\theta(\xi))}{d\xi^k}\right]_{\xi=0}.$$

As moments, these coefficients are characteristics of a probability distribution they are called cumulants or semiinvariants [30–32]. Cumulants unambiguously determine RQs if series (65) converges for all ξ. In this case, the set of cumulants $\chi_1, \chi_2, \ldots, \chi_n, \ldots$ can serve as the identical representation of a probability distribution. When moments are known cumulants can be found from the following relationships:

$$\chi_1 = m_1,$$
$$\chi_2 = m_2 - m_1^2 = D,$$
$$\chi_3 = m_3 - 3m_1 m_2 + 2m_1^2,$$
$$\chi_4 = m_4 - 3m_2^2 - 4m_1 m_3 + 12 m_1^2 m_2 - 6m_1^4,$$
$$\chi_5 = m_5 - 5m_4 m_1 - 10 m_2 m_3 + 20 m_1^2 m_3 + 30 m_1 m_2^2 - 60 m_1^3 m_2 + 24 m_1^5,$$
$$\ldots.$$

In turn moments are expressed through cumulants as

$$m_1 = \chi_1,$$
$$m_2 = \chi_2 + \chi_1^2,$$

Table 2. Moments and cumulants of the Kravchenko–Rvachev distribution

N	Moments	Cumulants
2	0.111	0.111
4	0.028	8.889×10^{-3}
6	9.793×10^{-3}	5.031×10^{-3}
8	5.082×10^{-3}	5.182×10^{-3}
10	1.919×10^{-3}	0.763

$$m_3 = \chi_3 + 3\chi_1\chi_2 + \chi_1^3,$$
$$m_4 = \chi_4 + 3\chi_2^2 + 4\chi_1\chi_3 + 6\chi_1^2\chi_2 + \chi_1^4,$$
$$m_5 = \chi_5 + 5\chi_1\chi_4 + 10\chi_2\chi_3 + 10\chi_1^2\chi_3 + 10\chi_1^3\chi_2 + 15\chi_1\chi_2^2 + \chi_1^5,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

If $p_\theta(x) = \frac{1}{b} up\left(\frac{x-m}{b}\right)$ then all of the odd moments and all of the odd cumulants are zeros. Let us present the numerical values of the first five even moments and cumulants (Table 2).

Note that the asymmetry and excess coefficients found above are cumulant coefficients [30–32] that describe the degree of deviation of a probabilistic distribution from the Gaussian function. Assume that there is a distribution $p_\theta(\xi)$ having all cumulants. Its CF can be represented as

$$F_\theta(\xi) = \exp\left\{\left(im\xi - \frac{D^2}{2}\xi^2\right)\left[1 + \sum_{k=3}^{\infty} \frac{\beta_k}{k!}(i\xi)^k\right]\right\}. \tag{66}$$

Coefficients β_k from (66) are the moments of distribution (48) that are determined for $\chi_{1,2} = 0$. These coefficients, called the quasi-moments of a distribution [43], are nonzero for non-Gaussian RQs only. The FT of (66) yields

$$p_\theta(x) = p_G(x) + \sum_{k=3}^{\infty} (-1)^k \frac{\beta_k}{k!} p_G^{(k)}(x). \tag{67}$$

Expresson (67) is called the Edgeworth series. It is the decomposition of an arbitrary RQ in derivatives of the Gaussian distribution [30–40]. With the first four even terms retained in (67) for distribution (48), we have

$$p_\theta(x) = p_G(x) + \frac{\chi_4}{4!} p_G^{(4)}(x) + \frac{\chi_6}{6!} p_G^{(6)}(x) + \frac{\chi_8 + 35\chi_4^2}{8!} p_G^{(8)}(x) + \frac{\chi_{10} + 210\chi_4\chi_6}{10!} p_G^{(10)}(x). \tag{68}$$

We present the simplest example of the application of the cumulant approach for constructing a 2D PDF in the case when RQs are coupled by correlation factor ρ only. In this situation the CF has the form

$$F_\theta(\xi,\eta) = \exp(-i\rho\xi\eta)\, F_\theta(\xi)\, F_\theta(\eta), \tag{69}$$

where $F_\theta(\xi)$ and $F_\theta(\eta)$ are the CFs of RQs with densities (64). Each of the CFs can be represented in form (66) or as

$$F_\theta(\xi) = \Psi_1(\xi)\exp\left[i\chi_{1\xi}\xi - \frac{\chi_{2\xi}}{2}\xi^2\right], \tag{70a}$$

$$F_\theta(\eta) = \Psi_1(\eta)\exp\left[i\chi_{2\eta}\eta - \frac{\chi_{2\eta}}{2}\eta^2\right]. \tag{70b}$$

Here, $\chi_{1\xi}, \chi_{2\xi}$ and $\chi_{1\eta}, \chi_{2\eta}$ are the first two cumulants of distribution (48) (see Table 2). With allowance for (70a) and (70b), 2D CF (69) can be represented in the form

$$F_\theta(\xi,\eta) = F_G(\xi,\eta)\,\Psi_1(\xi)\,\Psi_1(\eta). \tag{71}$$

Relationship (71) is the product of the CF of the Gaussian ensemble of two RQs $\{\delta_1, \delta_2\}$ and the Kravchenko–Rvachev CF of two RQs $\{\delta_1', \delta_2'\}$. The Fourier transform of (71) yields the PDF

$$p(x,y) = \iint_{\mathbf{R}^2} p_G(s,t)\, p_1(x-s)\, p_2(y-t)\, ds dt, \tag{72}$$

where

$$p_G(s,t) = \frac{1}{2\pi\sqrt{\chi_{2\xi}\chi_{2\eta} - \rho^2}}$$
$$\times \exp\left[\frac{\chi_{2\eta}(s-\chi_{1\xi})^2 - 2\rho(s-\chi_{1\xi})(t-\chi_{1\eta}) + \chi_{2\xi}(t-\chi_{1\eta})^2}{2(\chi_{2\xi}\chi_{2\eta} - \rho^2)}\right],$$

and $p_1(x)$ and $p_2(y)$ are the Kravchenko–Rvachev PDFs specified by their cumulants (65) and (67). Thus, we have obtained the 2D PDF of correlated RQs. When $\rho = 0$, (72) becomes (64).

F. Methods of Digital Simulation of Random Quantities with the Kravchenko–Rvachev Distribution

Mathematical models of radio signals and radio interferences are random processes that can be represented in the general form

$$s(t) = f\left(s_1(t, x_1, x_2, \ldots), s_2(t, x_1, x_2, \ldots), \ldots, \xi_1(t), \xi_2(t), \ldots\right), \tag{73}$$

where $t \in \mathbf{R}$ is the time, $s_1(t, x_1, x_2, \ldots)$ and $s_2(t, x_1, x_2, \ldots)$ are functions with random parameters, $\xi_1(t)$ and $\xi_2(t)$ are random processes (noises) with specified properties, and *f* is a certain function. Practically every oscillation observed at a certain point of a radio channel can be represented in form (73). The purpose of simulation of radio signals and radio interferences is to reproduce random processes of form (73) that mathematically describe radio signals and radio interferences at DSP devices. A process with a uniform PDF is the simplest for simulation. A random process with the Kravchenko–Rvachev distribution law can be obtained by summing weighted random numbers with a uniform PDF. Random processes with the Kravchenko–Rvachev distribution can also be simulated with the help of the following methods: simulation with the help of the gamma distribution [30–32], the method of a forming filter, the sliding summation method, and the method of canonical and noncanonical representations.

Atomic quasi-harmonic stationary processes. Let us present certain applications of the Kravchenko–Rvachev atomic distribution applied for investigation of a narrowband stationary noise. Consider a physical process of the form

$$\xi(t) = \rho(t)\cos[\omega_0 t + \varphi(t)]. \tag{74}$$

The PDF of process $\xi(t)$ has form (48). Random functions ξ, ρ, and φ are assumed stationary. The stationarity condition implies in particular that distribution $p(\varphi)$ is uniform:

$$p(\varphi) = 1/2\pi, \quad \varphi \in [-\pi, \pi]. \tag{75}$$

Let us find the CF of process (74). We have

$$F(x) = \int_0^\infty d\rho \int_{-\pi}^{\pi} d\varphi \exp(ix\rho\cos[\omega_0 t + \varphi])\, p(\rho,\varphi). \tag{76}$$

Here, $p(\rho,\varphi)$ is the joint PDF of ρ and φ that corresponds to distribution $p(\xi)$. Since $\varphi \in [-\pi, \pi]$, we can decompose $p(\rho,\varphi)$ in the Fourier series in variable φ

$$p(\rho,\varphi) = \sum_{k\in\mathbf{Z}} V_k(\rho)\exp(ik\varphi). \tag{77}$$

Let us represent factor $\exp(ix\rho\cos[\omega_0 t + \varphi])$ from (76) in an analogous form as

$$\exp(ix\rho\cos[\omega_0 t + \varphi]) = \sum_{l\in\mathbf{Z}} U_l(x\rho)\exp(il(\omega_0 t + \varphi)), \tag{78}$$

where $U_l(x\rho) = i^l J_l(x\rho)$ and $J_l(x\rho)$ is the *l*th-order Bessel function. The substitution of (78) and (77) into (76) and integration of the result with respect to φ yield

$$F(x) = 2\pi\sum_{k\in\mathbf{Z}} i^k \exp(ik\omega_0 t)\int_0^\infty V_k(\rho)\, J_k(x\rho)\, d\rho. \tag{79}$$

Since process $\xi(t)$ is assumed stationary, $F(x)$ is independent of time. This means that all $V_k(\rho)$ except $V_0(\rho)$ are zero, whence it follows that $p(\rho,\varphi) = V_0(\rho)$. Hence, $p(\varphi) = \text{const}$, which implies (75). Since equality (75) holds, $p(\xi)$ and $p(\rho)$ are unambiguously coupled. Let us modify (79) into the form

$$F(x) = \int_0^\infty p(\rho) J_0(x\rho) d\rho, \tag{80}$$

$$p(\rho, \varphi) = p(\rho) 1/2\pi. \tag{81}$$

Transforming (81) with the use of the new variables $\xi = \rho\cos[\omega_0 t + \varphi]$ and $\eta = \rho\sin[\omega_0 t + \varphi]$ and eliminating η via integration, we obtain

$$p(\xi) = \frac{1}{\pi}\int_0^\infty d\eta \left.\frac{p(\rho)}{\rho}\right|_{\rho=\sqrt{\xi^2+\eta^2}} = \frac{1}{\pi}\int_{|\xi|}^\infty d\rho \frac{p(\rho)}{\sqrt{\rho^2 - \xi^2}}. \tag{82}$$

Expression (80) which is the Hankel transformation yields

$$p(\rho) = \rho \int_0^\infty J_0(x\rho) F(x) x dx. \tag{83}$$

For the Gaussian distribution $p(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\left(-\frac{\xi^2}{2\sigma^2}\right)$ with the CF $F(x) = \exp\left(-\frac{x^2\sigma^2}{2}\right)$ the corresponding distribution for the envelope has the form $p(\rho) = \frac{\rho}{\sigma^2}\exp\left(-\rho^2/2\sigma^2\right)$. For Kravchenko–Rvachev distribution (48) at $b = 1$ and $m = 0$ the CF is

$$F(x) = \prod_{k=1}^\infty \operatorname{sinc}\left(\frac{x}{2^k}\right).$$

Figure 3 illustrates the comparison of $p(\rho)$ for the Gaussian and Kravchenko–Rvachev laws in the case of the corresponding normalization. The approach developed above can be used to describe random processes of a more general form such that

$$\xi(t) = \rho(t) F[\omega_0 t + \varphi(t)], \quad F_{\max} = 1, \\ \varphi \in [-\pi, \pi], \quad \rho > 0. \tag{84}$$

A random sequence of pulses can be represented in form (54). Then function $F(\cdot)$ describes the shape of a pulse that is not distorted by fluctuations, ρ and φ denoting random height and phase fluctuations of the pulse respectively.

Generalized PDFs based on the Kravchenko–Rvachev distribution. Finite distributions based on AFs $up(x)$ can be useful in practice. The numerical characteristics of the proposed distributions are summarized in Table 3 and shown in Fig. 4.

The Kravchenko–Poisson PDF (Fig. 4a) is

$$p(x) = \begin{cases} \frac{1}{c} up\left(\frac{x}{b}\right)\exp\left(-\frac{x}{a}\right) & \text{when } x \ge 0, \\ 0 \quad \text{when } x < 0, \end{cases} \tag{85}$$

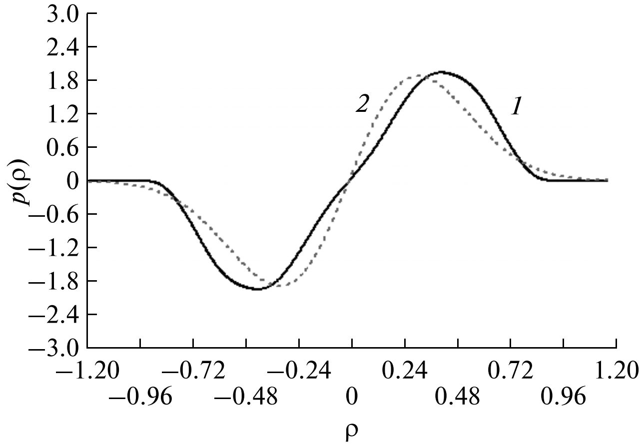

Fig. 3. (*1*) Kravchenko–Rvachev and (*2*) Gaussian distributions for envelope $p(\rho)$.

where $a, b, c \in \mathbf{R}$.

The Kravchenko–Gaussian PDF (Fig. 4b) is

$$p(x) = \frac{1}{\sqrt{2\pi}c}\exp\left(-\frac{x^2}{2a^2}\right)up\left(\frac{x}{b}\right), \quad a, b, c \in \mathbf{R}. \tag{86}$$

The Kravchenko–Rayleigh PDF (Fig. 4c) is

$$p(x) = \begin{cases} \frac{x}{c} up\left(\frac{x}{b}\right)\exp\left(-\frac{x^2}{2a^2}\right) & \text{when } x \ge 0, \\ 0 \quad \text{when } x < 0, \end{cases} \tag{87}$$

where $a, b, c \in \mathbf{R}$.

The Kravchenko–Cauchy PDF (Fig. 4d) is

$$p(x) = \frac{1}{c}\frac{\frac{a}{\pi}}{a^2 + x^2} up\left(\frac{x}{b}\right), \quad a, b, c \in \mathbf{R}. \tag{88}$$

Here, c is chosen such that $\int_{\mathbf{R}} p(x) dx = 1$.

The theory of random processes. As is known [30–40], random process $\xi(t)$ can be completely

Table 3. Moments of probability densities for various distribution functions (85)–(88) (with the parameter $a = b = 1$)

m_1	m_2	m_3	m_4	m_5
Kravchenko–Poisson				
0.095	0.035	0.016	0.008023	0.00443
Kravchenko–Gauss				
0	0.134	0	0.0013	0
Kravchenko–Rayleigh				
0.387	0.18	0.093	0.053	0.031
Kravchenko–Cauchy				
0	0.099	0	0.023	0

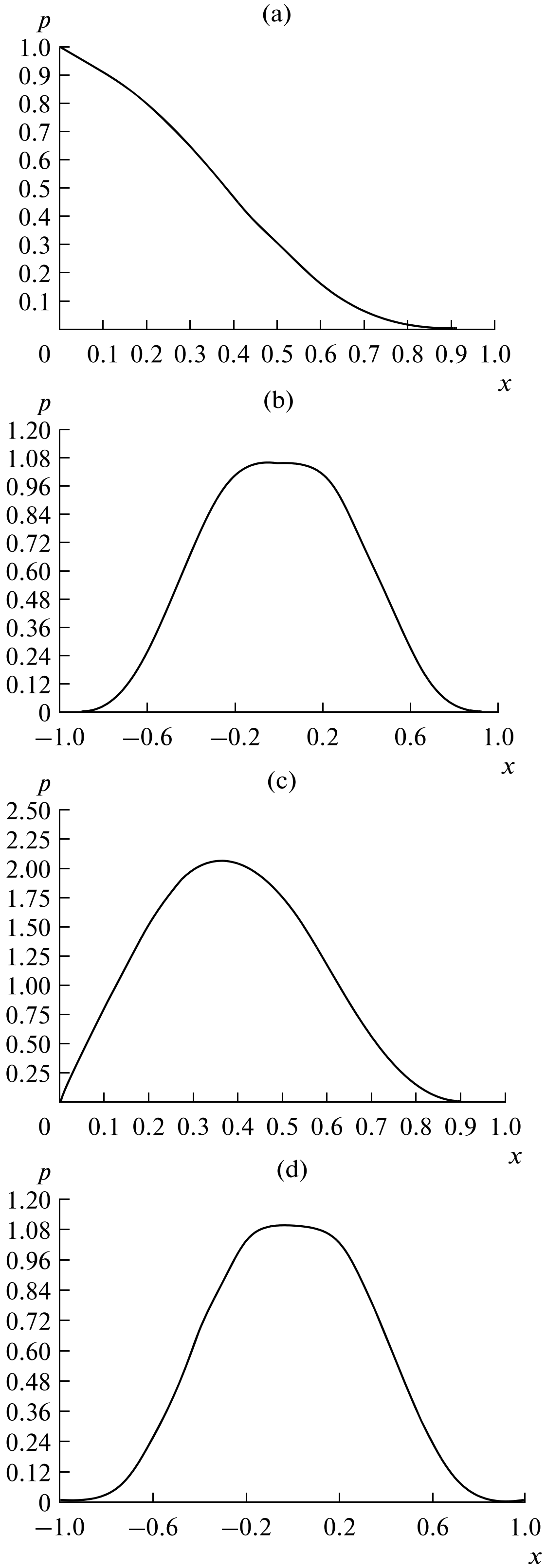

Fig. 4. (a) Kravchenko–Poisson, (b) Kravchenko–Gauss, (c) Kravchenko–Rayleigh, and (d) Kravchenko–Cauchy distribution functions.

determined by the set of PDFs of RQ ξ. The set includes 1D PDFs, i.e., $\forall t \in \mathbf{R}$:

$$W(x,t) = P\{\xi(t) < x\}, \tag{89}$$

where $P\{\xi(t) < x\}$ is the probability of the fact that at instant t the process takes a value less than x. For each pair of instants t_1 and t_2, the 2D PDF has the form

$$W(x_1,x_2,t_1,t_2) = P\{\xi(t_1) < x_1; \xi(t_2) < x_2\}. \tag{90}$$

The first and second order moments play a special role in the theory of random processes. This section of the theory of random processes is called the correlation theory. The correlation theory is extremely important for practice, because the correlation function rather completely characterizes the time structure of a random process. In addition the correlation function is rather easy to determine from experimental data. First and second order moments $a(t)$ and $\sigma^2(t)$ of a process stationary in the wide sense are independent of time and correlation function $\mathrm{B}(t_1,t_2)$ of such a process depends on the difference $\tau = t_1 - t_2$

$$a(t) = \text{const}, \quad \sigma^2(t) = \text{const}, \quad B(t_1,t_2) = B(\tau). \tag{91}$$

As a rule a process under study is represented by one realization or a small number of realizations. Therefore, the following ergodic property is used: time averaging is equivalent to averaging over the set of realizations,

$$a(t) = E\xi(t) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t)\,dt, \tag{92a}$$

$$\sigma^2 = E[\xi(t) - a]^2 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[\xi(t) - a]^2\,dt, \tag{92b}$$

$$B(\tau) = E[\xi(t-\tau)\xi(t)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\xi(t-\tau)\xi(t)\,dt. \tag{92c}$$

Below, we study stationary random processes. Energy spectrum $g(\omega)$ of a random process is determined from the expression

$$g(\omega) = \int_{\mathbf{R}} B(\tau)\exp(-i\omega\tau)\,d\tau. \tag{93}$$

The inversion formula

$$B(\tau) = \frac{1}{2\pi}\int_{\mathbf{R}} g(\omega)\exp(i\omega\tau)\,d\omega$$

is valid.

Estimation of the spectrum of a random process. One of the important problems involved in the study of random processes is as follows. There is one or several time-bounded realizations of a random process. It is necessary to characterize the process as a whole on the basis of these data. It is assumed that such a process (having an infinite number of realizations) can exist. If the process is studied within the framework of the correlation theory, then, it is necessary to find the correlation or spectral density function of a process represented by one or several realizations. A certain functional of available realizations is applied to determine the spectrum of the process.

Spectral estimation is considered to mean the procedure of spectrum determination, and the corresponding functional is referred to as an estimate [30–40]. Let us introduce the following notation: $g(\omega)$ is a spectrum, $\hat{g}(\omega)$ is the estimate of the spectrum, $B(\tau)$ is a correlation function, and $\hat{B}(\tau)$ is the estimate of the correlation function.

The estimation of the spectrum of a random process consists of three stages: measurement of sample values, construction of the estimating functional, and calculation of the estimate of a spectrum. When the problem of estimating the numerical parameters (e.g., the variance) of a random process is solved, the requirements imposed on the estimating functional can accurately be formulated. For example, it is necessary to ensure the minimum RMSD. When the spectral density, which is a function of frequency, is estimated, the requirements on the estimate are more severe. The error in spectrum estimation consists of two components: systematic and fluctuational. In every particular case the parameters of the estimating functional should be chosen such that the estimate is the best suitable for a problem to be solved.

Estimates of the spectral density and atomic functions. Let one or several time-bounded realizations $\xi_T(t)$ be known. Then the estimate is constructed with the use of the correlation function as the product of τ-shifted realizations that is time-averaged. The following expression is valid:

$$\hat{B}_1(\tau) = \frac{1}{T}\int_0^T \xi_T(t)\xi_T(t+\tau)dt, \quad |\tau| \le T, \tag{94}$$

where T is the duration realization $\xi_T(t)$.

Since estimating functional (94) differs from the definition of the correlation function, estimate $\hat{B}_1(t)$ equals $B_T(t)$ on the average only every estimation result randomly deviates from true values and the estimate is not defined for $|\tau| > T$.

The estimate of the correlation functional is often represented in the form

$$\hat{B}_2(\tau) = \frac{1}{T-|\tau|}\int_0^T \xi_T(t)\xi_T(t+\tau)dt = \frac{T}{T-|\tau|}\hat{B}_1(\tau), \quad |\tau| \le T. \tag{95}$$

Formula (95) can be represented in a more general form. Let us introduce a WF for example, $up(\tau)$ to obtain the following estimate:

$$\hat{B}(\tau) = up(\tau)\hat{B}_2(\tau). \tag{96}$$

Estimate (96) can be represented in a more general form if AF $up\left(\frac{t}{T}\right)$ with support $[-T, T]$ is introduced as WF $a(t)$. Then

$$\begin{aligned}\hat{B}(\tau) &= \frac{1}{c}\int_{\mathbf{R}} \xi(t)a(t)\xi(t+\tau)a(t+\tau)dt \\ &= \frac{1}{c}\int_{\mathbf{R}} \xi(t)up\left(\frac{t}{T}\right)\xi(t+\tau)up\left(\frac{t+\tau}{T}\right)dt,\end{aligned} \tag{97}$$

$$\xi_T(t) = \xi(t)a(t) = \xi(t)up\left(\frac{t}{T}\right),$$

where c is the normalizing factor having the meaning of the averaging interval length. For example $c = T - |\tau|$ for $\hat{B}_2(\tau)$.

From (97) we find the relationship between $g(\tau)$ and $a(\tau)$

$$E\hat{B}(\tau) = B(\tau)\frac{1}{c}\int_{\mathbf{R}} a(t)a(t+\tau)dt = g(\tau)B(\tau),$$

$$g(\tau) = \frac{1}{c}\int_{\mathbf{R}} a(t)a(t+\tau)dt. \tag{98}$$

Hence, WF $a(t)$ generates WF $g(\tau)$. When $a(t) = up(t/T)$ we find

$$\begin{aligned}g(t) &= \frac{1}{c}\int_{\mathbf{R}} up\left(\frac{t}{T}\right)up\left(\frac{t+\tau}{T}\right)dt\Bigg|_{l=\frac{t}{T}} \\ &= \frac{T}{c}\int_{\mathbf{R}} up(l)\,up\left(l+\frac{\tau}{T}\right)dl = \frac{T}{c}cup\left(\frac{\tau}{T}\right).\end{aligned}$$

Note that function $cup(\tau)$ us investigated in [5].

Let us estimate spectrum $\hat{g}(\omega)$ using the expression $\hat{g}(\omega) = \int_{\mathbf{R}} \hat{B}(\tau)\exp(-i\omega\tau)\,d\tau.$

The mean of estimate $E\hat{g}(\omega)$ equals the smoothed real spectrum

$$E\hat{g}(\omega) = \int_{\mathbf{R}} B(\tau)\exp(-i\omega\tau)\,d\tau = \int_{\mathbf{R}} g(\omega)Q(\omega-\tilde{\omega})d\tilde{\omega}, \tag{99}$$

where the spectrum of WF $g(\tau)$ serves as a smoothing function.

Taking into account (98) and (99) we have

$$\begin{aligned}E\hat{g}(\omega) &= \frac{T}{c}\int_{\mathbf{R}} B(\tau)cup\left(\frac{\tau}{T}\right)\exp(-i\omega\tau)\,d\tau \\ &= \int_{\mathbf{R}} g(\omega)\prod_{k=1}^{\infty}\operatorname{sinc}^2\left(\frac{\omega-\tilde{\omega}}{2^k}\right)d\tilde{\omega}.\end{aligned} \tag{100}$$

If

$$\hat{g}_1(\omega) = \int_{\mathbf{R}} B_1(\tau)\exp(-i\omega\tau)d\tau$$

Table 4. Operations performed with the help of weighting functions

Time function $\xi(t)$	Amplitude spectrum $c(\omega)$	Correlation function $B(\tau)$	Energy spectrum $g(\omega)$
Time weighting $a(t)$	Frequency smoothing $c(\omega) * A(\omega)$	Time shift weighting $B(\tau)g(\tau)$	Frequency smoothing $g(\omega) * Q(\omega)$
Time smoothing $\xi(t) * h(t)$	Frequency weighting $c(\omega)H(\omega)$	Time shift smoothing $B(\tau) * p(\tau)$	Frequency weighting

is assumed to be the initial estimate, the introduction of additional WF $g(\tau)$ smoothes the initial estimate of the spectrum

$$\hat{g}(\omega) = \int_{\mathbf{R}} \hat{g}_1(\tilde{\omega})Q(\omega - \tilde{\omega})d\tilde{\omega},$$

where $Q(\omega)$ is the spectrum of additional WF $g(\tau)$.

Let us summarize the above analysis. The correlation spectrum is estimated in the following sequence: correlation function $B_1(\tau)$ is estimated; next, WF $g(\tau)$ (e.g., $\frac{T}{c} cup\left(\frac{\tau}{T}\right)$) is introduced; and the estimate of the spectrum is found by applying the Fourier transform to the weighted estimate. In [14, 36] the filtered estimate of a spectrum is considered in more detail.

Weighting and smoothing in the spectral analysis based on atomic functions. In the discussion of the correlation estimate of a spectrum we have introduced WF $a(t)$ for a process realization, WF $g(\tau)$ for estimating the correlation function and corresponding smoothing function $Q(\omega)$ for estimating the spectrum.

Weighting functions applied in spectral analysis can be categorized into several types according to the character of operations performed with the help of these WFs. The basic operations performed with the help of WFs [5–21] are presented in Table 4.

Weighting functions $g(\tau)$ and $\beta(\tau)$ can be generated by WFs $a(t)$ and $h(t)$. Then

$$g(t) = \frac{1}{c}\int_{\mathbf{R}} a(t)a(t+\tau)\,dt, \quad Q(\omega) = \frac{1}{a}|A(\omega)|^2,$$

$$p(t) = \int_{\mathbf{R}} h(t)h(t+\tau)\,dt, \quad P(\omega) = |H(\omega)|^2.$$

In [5–21], it is shown that, when certain characteristics are to be optimized, the use of WFs based on AFs should be preferred to the use of classical WFs. The following Kravchenko–Rvachev weighting functions are applied most often:

$$a_1(t) = up(t), \quad a_2(t) = up(t) + 0.01up'(t),$$

$$a_3(t) = fup_1(3t/2)/fup_1(0), \quad a_4(t) = h_{3/2}(t),$$

$$a_5(t) = \Xi_2(t)/\Xi_2(0).$$

The main properties of WFs are presented in [5]. The flowchart that illustrates introducing WFs for estimating the correlation function and spectrum is displayed in Fig. 5.

4. INTERPOLATION OF STATIONARY RANDOM PROCESSES WITH ATOMIC FUNCTIONS

A. General Expressions for the Variance of the Error in the Stationary Random processes Interpolated from Discrete Sample Values

The functional block diagram of the system that discretizes and restores continuous signal $s(t)$ is displayed in Fig. 6. Here $s_\Delta(k)$ is a signal discretized with step Δt (formed from the original continuous signal by a switch device) and $\tilde{s}(t)$ is the restored signal. Discrete signal $s_\Delta(k)$ approximately represents original signal $s(t)$ with a certain error. Then the recovery of signal $s(t)$ from discrete process $s_\Delta(k)$ can be represented as the transmission of a sequence of Dirac δ functions through an interpolating filter with transient response $h(t)$ (see Fig. 6). Interpolated signal $\tilde{s}(t)$ has the form

$$\tilde{s}(t) = \sum_{k=-\infty}^{\infty} s_\Delta(k)h(t - k\Delta t), \tag{101}$$

where Δt is the discretization step and $s_\Delta(k) \equiv s[k\Delta t]$. The interpolation error

$$\Delta s(t) = s(t) - \tilde{s}(t) \tag{102}$$

is considered as the output signal of the circuit.

Let us present relationships for the correlation function, energy spectrum, and variance of error $\Delta s(t)$. Here interpolation error $\Delta s(t)$ is a stationary random process and its averaged correlation function has the form

$$R_\Delta(\tau) = \frac{1}{\Delta t}\int_0^{\Delta t} M\{\Delta s(t)\Delta s(t+\tau)\}\,dt. \tag{103}$$

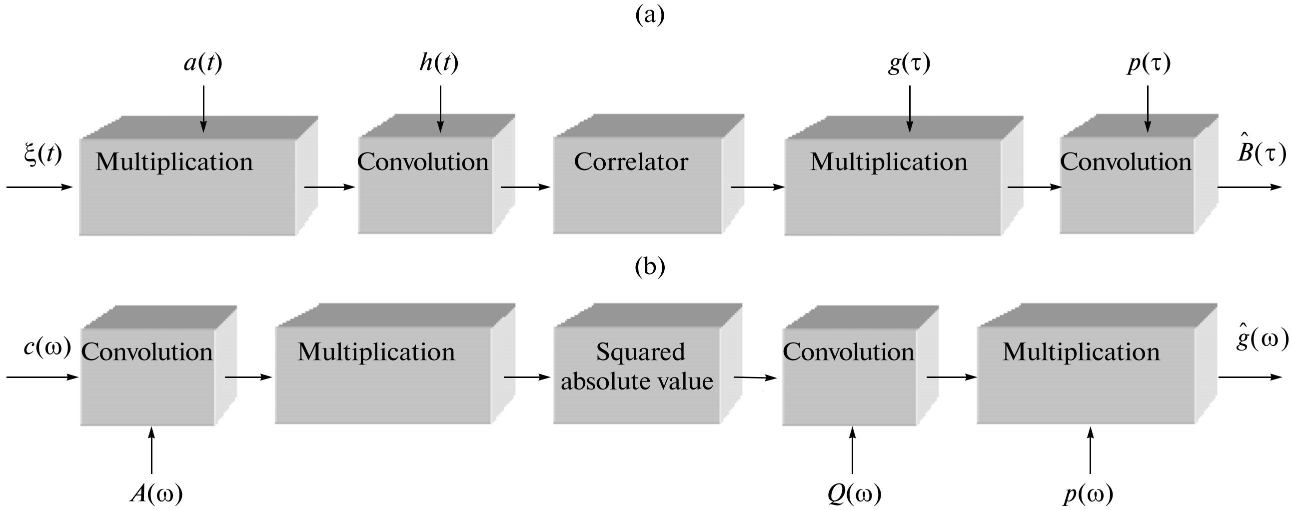

Fig. 5. Flowchart of introduction of the Kravchenko–Rvachev weighting and smoothing functions (windows) for estimating (a) correlation function $\hat{B}(\tau)$ and (b) the spectrum.

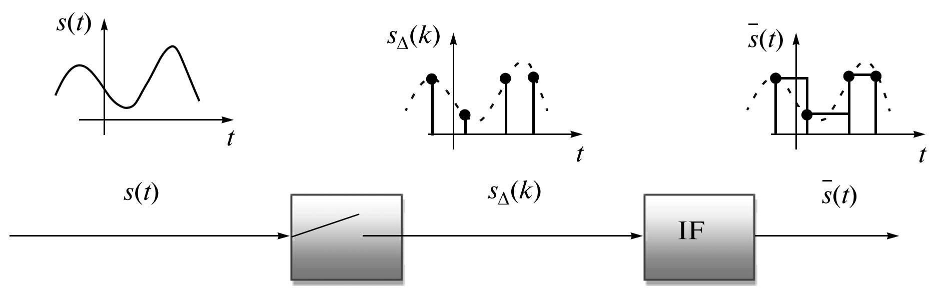

Fig. 6. Procedure of discretization and restoration of continuous signal $s(t)$.

Let $R(\tau)$ be the correlation function of the original random process. Then, the substitution of (101) and (102) into (103) yields the relationships between $R_\Delta(\tau)$ and $R(\tau)$ and between Δt and $h(t)$. After some algebra we obtain

$$R_\Delta(\tau) = R(\tau) + \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} R[k]\, g(\tau - k\Delta t) - \frac{1}{\Delta t}\big(R(\tau) * h(\tau) + R(\tau) * h(-\tau)\big), \qquad (104)$$

where the asterisk denotes convolution, $g(\tau) = h(\tau) * h(-\tau)$ and $R[k] = R(k\Delta t)$.

It is known [43] that the energy spectrum is the FT of the correlation function. Interpolation error $G_\Delta(\omega)$ for the energy spectrum can be expressed in terms of energy spectrum $G(\omega)$ of the original signal, FC $K(\omega)$ of the filter and discretization step Δt in the following general form:

$$G_\Delta(\omega) = G(\omega)\left[1 - \frac{2}{\Delta t}\operatorname{Re} K(\omega)\right] + \frac{1}{\Delta t}\Phi(\omega)\left|K(\omega)\right|^2, \qquad (105)$$

where

$$\Phi(\omega) = \sum_{k=-\infty}^{\infty} R[k]\exp(-i\omega\Delta t k) = \sum_{k=-\infty}^{\infty} G(\omega - 2\pi k/\Delta t).$$

We set $\tau = 0$ in (104) to obtain the expression for the interpolation error variance averaged over period Δt

$$\sigma_\Delta^2 = R_\Delta(0) = \sigma^2 + \frac{1}{\Delta t}\sum_{k=-\infty}^{\infty} R[k]\, g(k\Delta t) - \frac{2}{\Delta t}\int_{-\infty}^{\infty} R(\theta)\, h(\theta)\, d\theta,$$

where $\sigma^2 = R(0)$ is the variance of the original random process. The relative interpolation root-mean-square error (RMSE) $\Delta^2 = \sigma_\Delta^2/\sigma^2$ has the form

Table 5. Impulse and frequency characteristics $h(t)$ and $K(\omega)$ of interpolating filters

Interpolating filter	$h(t)$	$K(\omega)$
Zero-order element (nonsymmetric)	$\begin{cases}1, & t \in [0, \Delta t], \\ 0, & t \notin [0, \Delta t]\end{cases}$	$\Delta t \operatorname{sinc}\left(\frac{\omega \Delta t}{2}\right) \exp\left(\frac{j\omega \Delta t}{2}\right)$
Zero-order element (symmetric)	$\begin{cases}1, & t \in [-\Delta t/2, \Delta t/2], \\ 0, & t \notin [-\Delta t/2, \Delta t/2]\end{cases}$	$\Delta t \operatorname{sinc}\left(\frac{\omega \Delta t}{2}\right)$
First-order element	$\begin{cases}1 - \lvert t\rvert/\Delta t, & t \in [-\Delta t/2, \Delta t/2], \\ 0, & t \notin [-\Delta t/2, \Delta t/2]\end{cases}$	$\Delta t \operatorname{sinc}^2\left(\frac{\omega \Delta t}{2}\right)$
Ideal lowpass filter	$\frac{\sin(\omega_0 t)}{\omega_0 t}$	$\begin{cases}\Delta t, & \omega \in [-\omega_0, \omega_0], \\ 0, & \omega \notin [-\omega_0, \omega_0].\end{cases}$

$$\Delta^2 = 1 + \frac{1}{\Delta t}\sum_{k \in \mathbf{Z}} r[k]\, g(k\Delta t) - \frac{2}{\Delta t}\int_{-\infty}^{\infty} R(\theta)\, h(\theta)\, d\theta,$$

where $r(\theta) = R(\theta)/\sigma^2$ is the correlation factor of the original random process.

Interpolating filters. Let us determine the optimal FC of an interpolating filter. This characteristic should provide for the minimum interpolation RMSE. To this end, it is necessary to minimize the expression $\sigma_\Delta^2 = \frac{1}{\pi}\int_{-\infty}^{\infty} G_\Delta(\omega)\, d\omega$. Let us represent $G_\Delta(\omega)$ as

$$G_\Delta(\omega) = G(\omega)\left[1 - \frac{2}{\Delta t}X\right] + \frac{1}{\Delta t}\Phi(\omega)\left\{X^2 + Y^2\right\}, \quad (106)$$

where $X = \operatorname{Re} K(\omega)$ and $Y = \operatorname{Im} K(\omega)$ and find the minimum of expression (106). To do this, we have to solve the system of equations

$$\partial G_\Delta/\partial X = 0, \quad \partial G_\Delta/\partial Y = 0. \quad (107)$$

We obtain from (107)

$$\begin{cases} -\frac{2G}{\Delta t} + \frac{2\Phi}{\Delta t} X_{\text{opt}} = 0, \\ \frac{2\Phi}{\Delta t} Y_{\text{opt}} = 0 \end{cases}$$

then $X_{\text{opt}} = G/\Phi$ and $Y_{\text{opt}} = 0$.

Thus the optimal FC of the interpolating filter has the form

$$K_{\text{opt}}(\omega) = G(\omega)/\Phi(\omega). \quad (108)$$

Then the minimum error in the interpolation of a stationary random process can be represented as

$$\sigma_{\Delta\text{opt}}^2 = \frac{1}{\pi}\int_0^{\infty} G(\omega)\left[1 - \frac{G(\omega)}{\Delta t \Phi(\omega)}\right] d\omega.$$

It follows from (108) that

$$h(t) = \frac{1}{2\pi}\int_{\mathbf{R}} \exp(i\omega t)\frac{G(\omega)}{\Phi(\omega)}\, d\omega.$$

Since the optimal interpolating filter is difficult to implement in practice, we apply various interpolating functions (Table 5, Fig. 7).

Simulation. To perform simulation, we choose the specific form $R(\tau) = \exp(-\omega_0 |\tau|)$ of the correlation function of a random process, where $\omega_0 = \pi/2\Delta t = 10\ \text{s}^{-1}$, and the energy spectrum of the signal $G(\omega) = \frac{2\omega_0}{\omega_0^2 + \omega^2}$. In this case,

$$\Phi(\omega) = \frac{\sinh(\omega_0 \Delta t)}{\cosh(\omega_0 \Delta t) - \cos(\omega \Delta t)}.$$

Let us compare the normalized characteristics $G_\Delta(\omega)/G_\Delta(0)$ for various interpolating functions (see Table 5) calculated from formula (105) with the optimum dependences $G_{\Delta\text{opt}}(\omega)/G_{\Delta\text{opt}}(0)$ calculated from (109) (Figs. 8a–8d). It follows from the analysis of the behavior of the energy spectra of the interpolation error (see Figs. 8a–8d) that the first-order element is the closest to the optimal case. In order to numerically estimate the results, we calculate relative interpolation RMSE Δ_{1-4}^2 from (106) and the relative RMSD of the functions $\frac{G_{\Delta 1-4}(\omega)}{G_{\Delta 1-4}(0)}$ from the function $\frac{G_{\Delta\text{opt}}(\omega)}{G_{\Delta\text{opt}}(0)}$ according to the formula

$$\delta_{1-4} = \frac{\int_0^{\infty}\left|\frac{G_{\Delta 1-4}(\omega)}{G_{\Delta 1-4}(0)} - \frac{G_{\Delta\text{opt}}(\omega)}{G_{\Delta\text{opt}}(0)}\right|^2 d\omega}{\int_0^{\infty}\left|\frac{G_{\Delta\text{opt}}(\omega)}{G_{\Delta\text{opt}}(0)}\right|^2 d\omega}.$$

The comparison of interpolation RMSEs Δ_{1-4}^2 and RMSDs δ_{1-4} for various recovery schemes is illustrated in Table 6. It follows from these data that the first-order element is the best in both of the characteristics and the zero-order nonsymmetric element is the

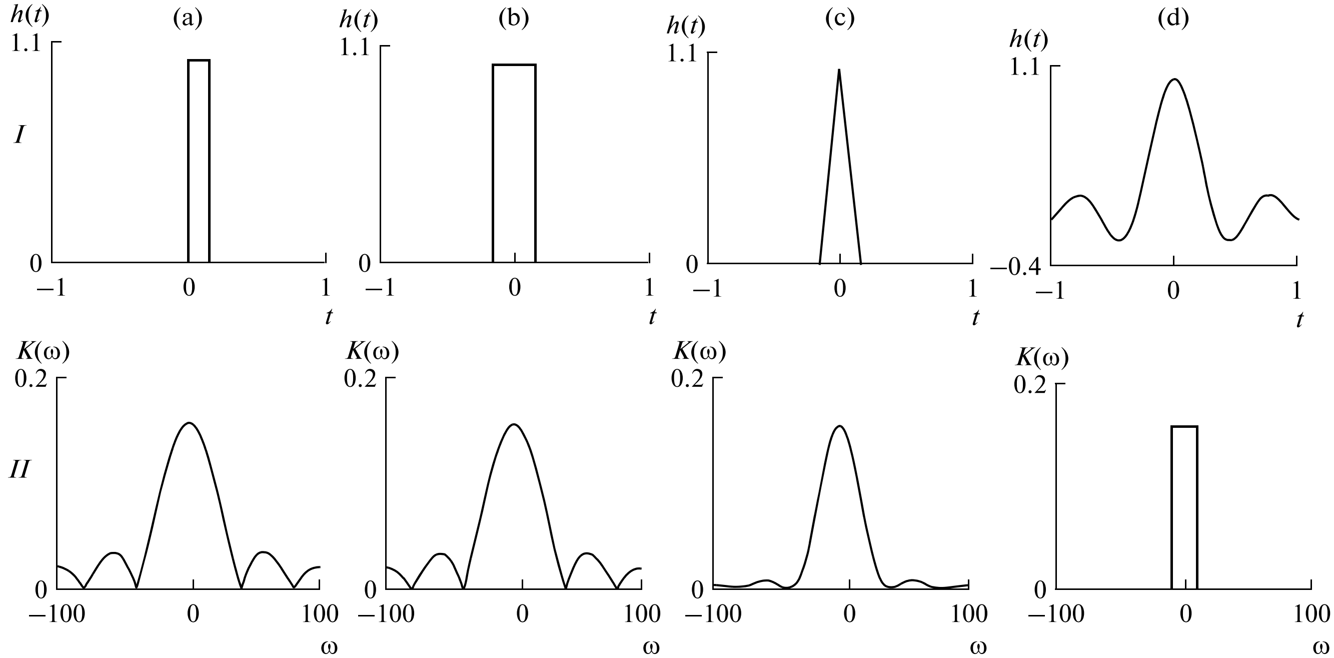

Fig. 7. (*I*) Impulse and (*II*) frequency characteristics of interpolating filters: (a) a zero-order nonsymmetric element, (b) a zero-order symmetric element, (c) a first-order element, and (d) a lowpass filter.

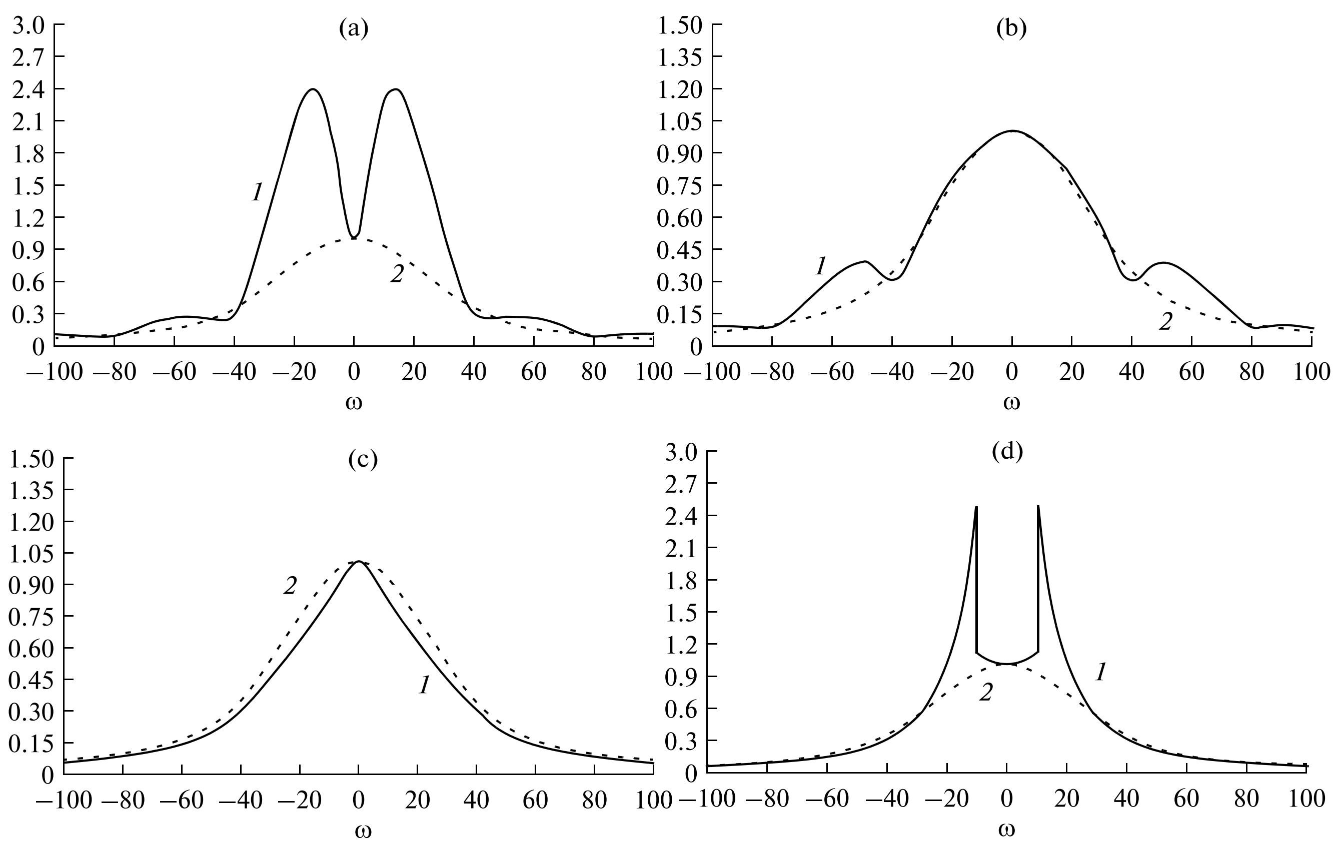

Fig. 8. Comparison of (curve *1*) the normalized spectra $G_{\Delta}(\omega)/G_{\Delta}(0)$ of the process under study and (curve *2*) the optimal case $G_{opt}(\omega)/G_{opt}(0)$ for zero-order (a) nonsymmetric and (b) symmetric elements, (c) a first-order element, and (d) a Kotelnikov ideal lowpass filter.

Table 6. Comparison of interpolation RMSEs and RMSDs with the help of the Δ^2_{1-4} and δ_{1-4} methods

Interpolating filter	Δ^2	δ
Zero-order element		
nonsymmetric	0.97467	1.55177
symmetric	0.58509	0.02472
First-order element	0.45635	0.01476
Ideal lowpass filter	0.61834	0.32833

Table 7. Comparison of interpolation RMSEs and RMSDs Δ^2 and δ for filters with FC (109)

Interpolation technique	$H_{2,2}(\omega)$	$H_{3,2}(\omega)$	$H_{2,3}(\omega)$	$H_{3,3}(\omega)$
Δ^2	**0.49651**	0.52469	**0.48918**	0.52157
δ	**0.00352**	0.00646	**0.00431**	0.00582

Here and in Table 8, the best results are in bold.

worst. Let us apply the interpolation methods based on the theory of AFs [5–19].

B. Interpolation of Stationary Random Processes with Atomic Functions

As interpolating functions, we use AFs $h_a(x)$ and $fup_n(x)$. Let us introduce the following notation:

$$H_{a,N}(\omega) = \Delta t \prod_{i=1}^{N} \operatorname{sinc}\left(\frac{\omega \Delta t}{2a^{i-1}}\right), \tag{109}$$

$$F_{a,n,N}(\omega) = \Delta t \operatorname{sinc}^n\left(\frac{\omega \Delta t}{2}\right) \prod_{i=1}^{N} \operatorname{sinc}\left(\frac{\omega \Delta t}{2a}\right). \tag{110}$$

Relationships (109) and (110) are scaled FTs of AFs $h_a(x)$ and $fup_n(x)$ that are represented analytically as [5]

$$h_a(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \prod_{i=1}^{\infty} \operatorname{sinc}\left(\frac{\omega}{a^i}\right) \exp(j\omega x)\, d\omega,$$

$$fup_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{sinc}^n\left(\frac{\omega}{2}\right) \prod_{i=1}^{\infty} \operatorname{sinc}\left(\frac{\omega}{2^i}\right) \exp(j\omega x)\, d\omega.$$

Relationship (110) becomes (109) when $n = 0$. Let us calculate interpolation RMSE Δ^2 and δ for the case when the FCs of interpolating filters are specified by expressions (109) (Table 7) and (110) (Table 8). It follows from the data presented in these tables that filters with FCs $H_{2,2}(\omega)$ and $H_{2,3}(\omega)$ exhibit the best characteristics in class $H_{a,N}(\omega)$ and that $F_{2,2,1}(\omega)$ and $F_{3,2,1}(\omega)$ exhibit the best characteristics in class $F_{a,n,N}(\omega)$. Note that, in filter class $H_{a,N}(\omega)$, relative RMSD δ can be reduced by an order of magnitude as compared to class $F_{a,n,N}(\omega)$ and to the linear interpolation; simultaneously, Δ^2 is deteriorated. Let $G_{\Delta a,N}(\omega)$ denote the energy spectrum of the interpolation error for filters from class $H_{a,N}(\omega)$. The comparison of the dependences $G_{\Delta a,N}(\omega)/G_{\Delta a,N}(0)$ determined from Tables 7 and 8 and the dependences $\frac{G_{\Delta\mathrm{opt}}(\omega)}{G_{\Delta\mathrm{opt}}(0)}$, which are the best in this class, is illustrated in Figs. 9a and 9b. Note that, at frequencies that are low (high) as compared to ω_0, the filter with FC $H_{3,2}(\omega)$ ($H_{2,3}(\omega)$) most closely approaches the optimal one.

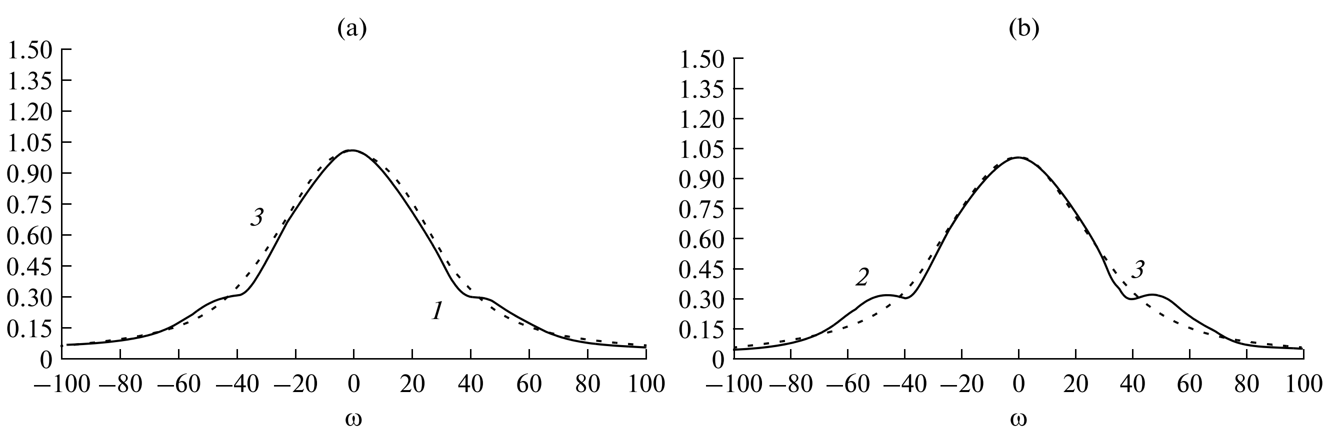

Fig. 9. Comparison of the characteristics (curve *1*) $\frac{G_{\Delta 2,2}(\omega)}{G_{\Delta 2,2}(0)}$ and (curve *2*) $\frac{G_{\Delta 3.2}(\omega)}{G_{\Delta 3.2}(0)}$ with (curve *3*) the optimal method $\frac{G_{\Delta\mathrm{opt}}(\omega)}{G_{\Delta\mathrm{opt}}(0)}$.

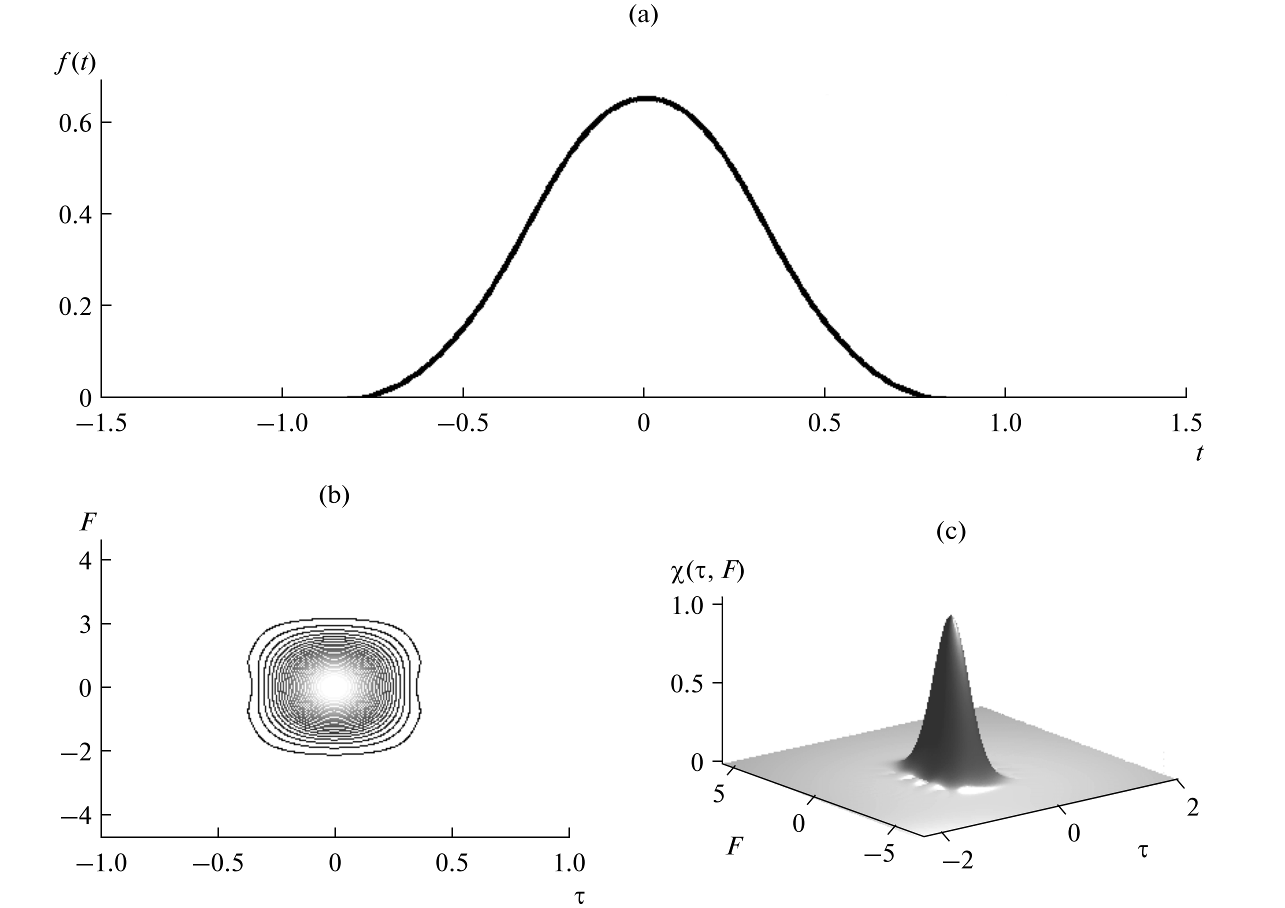

Fig. 10. (a) View of signal, (b) equipotential line of uncertainty function, and (c) the uncertainty surface of the Kravchenko–Gauss function ($g = 1.0$, $a = 2$, and $b = 1$).

5. A NEW CLASS OF PROBABILISTIC WEIGHTING FUNCTIONS IN DIGITAL SIGNAL AND IMAGE PROCESSING

A. Kravchenko–Cauchy–Gauss Probabilistic Weighting Functions

In [30–32], a new class of Kravchenko probabilistic distributions based on AFs $up(x)$ and $h_a(x)$ is proposed and justified. Using these results, let us consider the Kravchenko–Cauchy functions

$$f_1(x;a,b,g) = \frac{g}{c\pi\left(g^2 + x^2\right)} h_a\left(\frac{x}{b}\right) \tag{111}$$

and the Kravchenko–Gauss functions

$$f_2(x;a,b,g) = \frac{1}{\sqrt{2\pi}c} \exp\left(-\frac{x^2}{2g^2}\right) h_a\left(\frac{x}{b}\right), \tag{112}$$

where a, b, c, and g are real constants.

Taking into account the basic properties of AF $h_a(x)$ [5] we can formulate the following Theorem.

Theorem 7. The Kravchenko–Cauchy function of form (111) solves the following functional–differential equation with a shifted argument:

$$\frac{d}{dx} f(x) = \frac{1}{g^2 + x^2}\left\{-2xf(x) + \frac{a^2}{2b}\left[f(ax+1) \times \left(g^2 + (ax+1)^2\right) - f(ax-1)\left(g^2 + (ax-1)^2\right)\right]\right\}. \tag{113}$$

The proof of this theorem is considered in detail in [32].

Table 8. Comparison of interpolation RMSEs and RMSDs Δ^2 and δ for filters with FC (110)

Interpolation technique	$F_{2,2,1}(\omega)$	$F_{3,2,1}(\omega)$	$F_{2,3,1}(\omega)$	$F_{3,3,1}(\omega)$
Δ^2	**0.45710**	**0.45653**	0.46628	0.46500
δ	**0.01531**	**0.01513**	0.01192	0.01246

Table 9. Physical characteristics of a function ($g = 1$)

a	b	γ_2^*	γ_4^*	CG	ENB	SLL_{max}, dB	SAM, dB	TL_{max}
Kravchenko–Cauchy								
2	1	2.765	1.412	0.455	1.726	–26.666	1.068	3.440
3	2	2.500	1.357	0.582	1.415	–20.198	1.590	3.098
4	3	2.462	1.308	0.640	1.298	–18.788	1.904	3.038
5	4	2.500	1.417	0.672	1.237	–18.367	2.112	3.036
Kravchenko–Gauss								
2	1	2.812	1.437	0.474	1.683	–24.908	1.121	3.381
3	2	2.615	1.462	0.616	1.380	–18.647	1.680	3.080
4	3	2.500	1.417	0.682	1.268	–17.117	2.020	3.052
5	4	2.333	1.333	0.720	1.210	–16.573	2.246	3.072
Kravchenko–Kotelnikov–Cauchy $(M = 1,\ k = 2, \Delta = 1)$								
2	1	2.947	1.421	0.389	1.964	–30.788	0.834	3.764
3	2	2.812	1.437	0.463	1.667	–25.824	1.144	3.364
4	3	2.800	1.400	0.487	1.563	–26.959	1.300	3.241
5	4	2.733	1.400	0.498	1.515	–28.793	1.387	3.191
Kravchenko–Kotelnikov–Gauss $(M = 1,\ k = 2, \Delta = 1)$								
2	1	2.842	1.368	0.402	1.918	–29.138	0.871	3.699
3	2	2.687	1.375	0.483	1.624	–23.945	1.202	3.309
4	3	2.667	1.400	0.511	1.520	–24.381	1.372	3.190
5	4	2.786	1.429	0.523	1.470	–25.479	1.468	3.142
Kravchenko–Kotelnikov–Cauchy $(M = 3,\ k = 2, \Delta = 1)$								
2	1	2.950	1.400	0.374	2.028	–32.034	0.784	3.855
3	2	2.706	1.353	0.453	1.694	–26.605	1.110	3.398
4	3	2.867	1.467	0.481	1.578	–27.637	1.277	3.258
5	4	2.800	1.400	0.493	1.525	–29.387	1.370	3.202
Kravchenko–Kotelnikov–Gauss $(M = 3,\ k = 2, \Delta = 1)$								
2	1	2.947	1.421	0.386	1.982	–30.430	0.818	3.789
3	2	2.750	1.437	0.472	1.650	–24.661	1.166	3.341
4	3	2.667	1.400	0.504	1.534	–24.923	1.347	3.206
5	4	2.786	1.429	0.518	1.480	–25.909	1.450	3.151

$\gamma_2^* = \gamma_2/\gamma_3$ is the relative position of the first zero of the AFC (γ_3 is the half-power width of the AFC); $\gamma_4^* = \gamma_4/\gamma_3$ is the relative –6-dB-level width of the AFC, CG is the coherent gain, ENB is the equivalent noise bandwidth, SLL_{max} is the maximum sidelobe level, SAM is spurious amplitude modulations, and TL_{max} is the maximum transform loss.

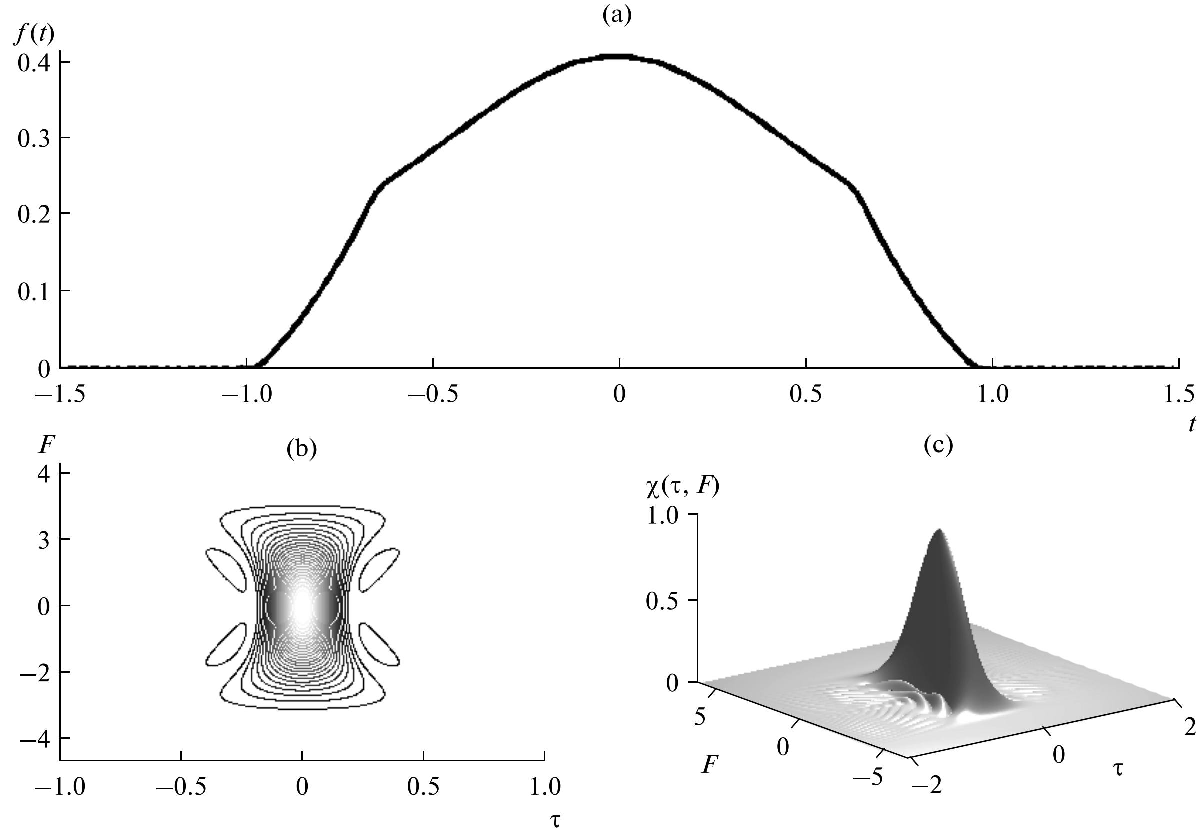

Fig. 11. View of signal (a), equipotential line of uncertainty function (b), uncertainty surface (c) of Kravchenko–Gauss function ($g = 1.0$, $a = 5$, $b = 4$).

With the notation

$$A(x) = \frac{1}{g^2 + x^2}, \quad Q_1(x) = \frac{a^2}{2b}\left(g^2 + (ax+1)^2\right), \quad Q_2(x) = \frac{a^2}{2b}\left(g^2 + (ax-1)^2\right), \tag{114}$$

we obtain

$$\frac{d}{dx} f(x) = A(x) \times \{-2xf(x) + f(ax+1)Q_1(x) - f(ax-1)Q_2(x)\}. \tag{115}$$

The segment $x \in \left[-\frac{b}{a-1}; \frac{b}{a-1}\right]$ is the support of function $f(x;a,b,g)$. If $b = a - 1$, we obtain the following particular case: $f(x;a,b,g) = 0 \ \forall \ x \notin [-1; 1]$.

B. The Physical Characteristics of Probabilistic Distributions

Consider the following probabilistic distributions:

(i) Kravchenko–Cauchy distribution (111),

(ii) Kravchenko–Gauss distribution (112),

(iii) Kravchenko–Kotelnikov–Cauchy distribution

$$f_3(x;a,b,g,M,k,\Delta) = \frac{g}{c\pi\left(g^2 + x^2\right)} h_a\left(\frac{x}{b}\right) \prod_{j=1}^{M} \operatorname{sinc} \frac{\pi x}{\Delta k^{j-1}}, \tag{116}$$

and

(iv) Kravchenko–Kotelnikov–Cauchy distribution

$$f_4(x;a,b,g,M,k,\Delta) = \frac{1}{\sqrt{2\pi}c} \exp\left(-\frac{x^2}{2g^2}\right) h_a\left(\frac{x}{b}\right) \prod_{j=1}^{M} \operatorname{sinc} \frac{\pi x}{\Delta k^{j-1}}, \tag{117}$$

where a, b, c, g, M, k, and Δ are real constants.

The modified characteristics investigated in [14] are used for physical analysis of the functions. The numerical values are summarized in Table 9.

C. Physical Analysis of Frequency–Time Properties of Probabilistic Functions

Consider frequency–time distributions of new probabilistic functions (111), (112), (116), and (117). The simulation and physical analysis of the results obtained have shown that the proposed functions can be widely used in various radiophysical applications.

For example, they can be used as pulses in problems of remote sensing of inhomogeneous media. Let us determine the form of the uncertainty function (UF)

$$\chi(\tau, F) = \int_{-\infty}^{\infty} f\left(t + \frac{\tau}{2}\right)\tilde{f}\left(t - \frac{\tau}{2}\right)\exp(i2\pi Ft)\,dt \qquad (118)$$

which behavior is presented in Figs. 10 and 11 for signals (111) and (112) for various parameters.

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REFERENCES

1. V. L. Rvachev, *Geometric Applications of Boolean Algebra* (Tekhnika, Kiev, 1967) [in Russian].
2. V. L. Rvachev, *Methods of Boolean Algebra in Mathematical Physics* (Naukova Dumka, Kiev, 1974) [in Russian].
3. V. L. Rvachev, *Theory of R-functions and Its Some Applications* (Naukova Dumka, Kiev, 1982) [in Russian].
4. V. L. Rvachev and V. A. Rvachev, *Nonclassical Methods of the Approximation Theory in Boundary Value Problems* (Naukova Dumka, Kiev, 1979) [in Russian].
5. V. F. Kravchenko, *Lectures on the Theory of Atomic Functions and Their Some Applications* (Radiotekhnika, Moscow, 2003) [in Russian].
6. V. F. Kravchenko and M. A. Basarab, *Boolean Algebra and Methods of Approximation in the Boundary Value Problems of Electromagnetics*, Ed. by V.F. Kravchenko (Fizmatlit, Moscow, 2004) [in Russian].
7. E. G. Zelkin, V. F. Kravchenko, and V. I. Gusevskii, *Approximation Design Methods in the Antenna Theory* (Sains-Press, Moscow, 2005).
8. V. F. Kravchenko and V. L. Rvachev, *Boolean Algebra, Atomic Functions and Wavelets in Physical Applications* (Fizmatlit, Moscow, 2006) [in Russian].
9. *Digital Signal and Image Processing in Radiophysical Applications*, Ed. by V. F. Kravchenko (Fizmatlit, Moscow, 2007) [in Russian].
10. V. K. Volosyuk and V. F. Kravchenko, *Statistical Theory of Radiotechnical Systems of Remote Sensing and Radar*, Ed. by V.F. Kravchenko (Fizmatlit, Moscow, 2008) [in Russian].
11. V. F. Kravchenko, H. M. Perez-Meana, and V. I. Ponomaryov, *Adaptive Digital Processing of Multidimentional Signals with Applications* (Fizmatlit, Moscow, 2009).
12. V. F. Kravchenko, O. S. Labun'ko, A. M. Lerer, and G. P. Sinyavskii, *Computing Methods in Modern Radio Physics* (Fizmatlit, Moscow, 2009) [in Russian].
13. O. V. Lazorenko and L. F. Chernogor, *Ultrawideband Signals and Processes* (V.N. Karazin Kharkov: Nats. Univ., Kharkov, 2009) [in Russian].
14. V. F. Kravchenko, O. V. Kravchenko, V. I. Pustovoit, and D. V. Churikov, Fiz. Osn. Priborostroen., Spec. Iss. p. 3 (2011).
15. V. K. Volosyuk, Yu. V. Gulyaev, V. F. Kravchenko, B. G. Kutuza, V. V. Pavlikov, and V. I. Pustovoit, J. Commun. Technol. Electron. **59**, p. 97 (2014).
16. V. F. Kravchenko, in *Radar and Radio Communication (Proc. 16th Int. Conf., Firsanovka, Moscow Reg., Nov. 11–16, 2008)* (MEI, Moscow, 2008), p. 463.
17. V. F. Kravchenko, O. V. Kravchenko, V. I. Pustovoit, and D. V. Churikov, in *Proc. 67th Sci. Session of the Popov Society Dedicated to the Radio Day, Moscow, May 16–17, 2012* (A.S. Popov RNTORES, Moscow, 2012), pp. III–VIII.
18. V. F. Kravchenko, in *Radiation and Scattering of Electromagnetic Waves (Proc. Int. Sci. Conf., Taganrog, Russia, 27 June–1 July, 2009* (Yuzhn. Federal. Univ., Taganrog, 2009), p. 24.
19. V. F. Kravchenko, in *Proc. Int. Conf. on Antenna Theory and Techniques, Lviv, Oct. 6–9, 2009* (Lviv. Politechnic Nat. Univ., Lviv, 2009), p. 3.
20. V. A. Rvachev, Usp. Mat. Nauk **45**, 77 (1990).
21. V. M. Kolodyazhnyi and V. A. Rvachev, Kibern. Sistemn. Anal., No. 6, p. 155 (2007).
22. V. F. Kravchenko and A. R. Safin, Elektromagn. Volny Elektron. Sist. **13** (12), p. 31 (2008).
23. V. F. Kravchenko and A. V. Yurin, J. Commun. Technol. Electron. **58**, p. 945 (2013).
24. V. F. Kravchenko, V. I. Pustovoit, and D. V. Churikov, J. Commun. Technol. Electron. **57**, 1050 (2012).
25. V. F. Kravchenko and D. V. Churikov, in *Proc. Progress in Electromagnetics Research Symp. (PIERS 2009), Moscow, Aug. 18–21, 2009* (The Electromagnetics Academy, Cambridge, USA, 2009), p. 241.
26. O. V. Kravchenko and A. R. Safin, in *Proc. Progress in Electromagnetics Research Symp. (PIERS 2009), Moscow, Aug. 18–21, 2009* (The Electromagnetics Academy, Cambridge (USA), 2009), p. 249.
27. V. F. Kravchenko, O. V. Kravchenko, A. R. Safin, in *Proc. 2010 Int. Symp. on Physics and Engineering of Microwaves, Millimeter and Submillimeter Waves (MSMW), Kharkov, Jun. 21–26, 2010* (IEEE, New York, 2010), MSMW.2010.5546004.
28. V. F. Kravchenko and D. V. Churikov, in *Proc. Int. Conf. "Days on Diffraction (DD-2012)", St. Petersburg, May 28–June 1, 2012* (IEEE, New York, 2012), p. 158.
29. V. F. Kravchenko and A. V. Yurin, Am. Inst. Phys. Conf. Proc. **1106**, 18 (2009).
30. V. F. Kravchenko, O. V. Kravchenko, and A. R. Safin, in *Proc. Int. Conf. "Days on Diffraction (DD-2009)", St. Petersburg, May 26–29, 2009* (IEEE, New York, 2009), p. 26.
31. V. F. Kravchenko, O. V. Kravchenko, and A. R. Safin, Usp. Sovremen. Radioelektron., No. 5, p. 23 (2009).
32. V. F. Kravchenko, O. V. Kravchenko, A. R. Safin, and D. V. Churikov, Elektromagn. Volny Elektron. Sist. **14** (9), p. 31 (2009).
33. V. F. Kravchenko and D. V. Churikov, J. Meas. Sci. Instrum. **4** (3), p. 231 (2013).
34. V. F. Kravchenko, V. I. Pustovoit, and D. V. Churikov, Dokl. Phys. **56**, p. 471 (2011).
35. V. F. Kravchenko, V. I. Pustovoit, and D. V. Churikov, Elektromagn. Volny Elektron. Sist. **16** (8), 44 (2011).

36. V. F. Kravchenko and D. V. Churikov, in *Proc. Int. Conf. "Days on Diffraction (DD-2012)", St. Petersburg, May 28–June 1, 2012* (IEEE, New York, 2012), p. 152.
37. V. F. Kravchenko, Ya. Yu. Konovalov, and D. V. Churikov, in *Proc. Progress in Electromagnetics Research Symp. (PIERS 2012), Moscow, Aug. 19–23, 2012.* (Electromagnetics Academy, Cambridge (USA), 2012), p. 45.
38. V. F. Kravchenko, O. V. Kravchenko, and A. R. Safin, in *Proc. Progress in Electromagnetics Research Symp. (PIERS 2009), Moscow, Aug. 18–21, 2009* (Electromagnetics Academy, Cambridge, USA, 2009), p. 237.
39. V. F. Kravchenko, O. V. Kravchenko, and A. R. Safin, in *Proc. Int. Conf. on Antenna Theory and Techniques, Lviv, Oct. 6–9, 2009* (Lviv. Politechnic Nat. Univ., Lviv, 2009), p. 66.
40. V. F. Kravchenko and D. V. Churikov, in *Proc. Int. Conf. "Days on Diffraction (DD-2011)", St. Petersburg, May 30–June 3, 2011* (IEEE, New York, 2011), p. 78.
41. Q. Jiang and S. L. Lee, Adv. Comput. Math. **7**, p. 383 (1997).
42. J. Shen, Methods Appl. Anal. **5**, 283 (1998).
43. R.Q. Jia, S. L. Lee, and A. Sharma, Proc. Am. Math. Soc. **126**, 729 (1998).
44. W. C. Lang, Int. J. Math. and Math. Sci. **2004**, p. 3435 (2004).
45. H. Gotovac, V. Cvetkovic, and R. Andricevic, Adv. Water Resour. **32**, 885 (2009).
46. V. A. Kotel'nikov, *Collected Works,* Vol. 1: *Radiophysics, Informatics, Telecommunications* (Fizmatlit, Moscow, 2008) [in Russian].
47. I. S. Ryzhak, *Selected Problems of the Circuit Theory and Signal Processing* (BLOK-Inform-Ekspress, Moscow, 2003) [in Russian].
48. A. Papoulis, *Systems and Transforms with Applications in Optics* (McGraw-Hill, New York, 1968; Mir, Moscow, 1971).

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