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COMPUTER SCIENCE

A New Class of WA-Systems of Kravchenko–Rvachev Functions

Academician **Yu. V. Gulyaev**[a], **V. F. Kravchenko**[a], and Academician **V. I. Pustovoit**[b]

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In this paper, we suggest and substantiate constructions of a new class of WA-systems of Kravchenko–Rvachev functions [1, 2] based on atomic functions (AFs) and using the ideas and results of [1–7]. In the first part, we describe an algorithm for constructing WA-systems of functions, and in the second part, we perform a computational study of a new family of complex wavelets based on AFs and possessing a number of useful properties. These wavelets and their Fourier images are determined by analytical relations.

CONSTRUCTION OF KRAVCHENKO–RVACHEV WAVELET FUNCTIONS

According to [1–3], the construction of a wavelet function ψ is determined by the requirement that a large number of the coefficients $\langle f, \psi_{j,n}\rangle$ must be close to zero. This number depends mainly on the regularity of the function f, the number of zero moments of ψ, and the size of the support of f. If f is a regular function and ψ has sufficiently many zero moments, then the wavelet coefficients $\langle f, \psi_{j,n}\rangle$ are small if the scale 2^j is small. In this case, ψ has p zero moments

$$\int_{-\infty}^{+\infty} t^k\psi(t)dt = 0, \quad \text{where} \quad 0 \le k < p. \qquad (1)$$

Theorem 1. *Suppose that* ψ *and* φ *are a wavelet and a scaling function generating orthogonal bases. If* $\psi(\omega)$ *and* $\varphi(\omega)$ *are p-times continuously differentiable at the frequency* $\omega = 0$, *then the following assertions are valid*:

the wavelet ψ *has p zero moments*;

$\psi(\omega)$ *and its first* $p - 1$ *derivatives vanish at* $\omega = 0$;

$h(\omega)$ *and its first* $p - 1$ *derivatives vanish at* $\omega = \pi$.

For any $0 \le k < p$ $q_k(t) = \sum_{n=-\infty}^{+\infty} n^k\varphi(t-n)$ is a polynomial of degree k.

The size of the support. If the function f has a singularity at t_0 and t_0 is inside the support of $\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-2^jn}{2^j}\right)$, then the amplitude of $\langle f, \psi_{j,n}\rangle$ may be large. If ψ has compact support of size K, then each level 2^j contains K wavelets $\psi_{j,n}$, whose supports contain the point t_0. To reduce the number of large coefficients, it is necessary to decrease the support of ψ. The following compact support theorem is valid [1, 3].

Theorem 2. *A scaling function* φ *is compactly supported if and only if so is h and the supports of* φ *and h are equivalent. If the supports of h and* φ *are equal to* $[N_1, N_2]$, *then the support of* ψ *equals* $\left[\frac{N_1 - N_2 + 1}{2}, \frac{N_2 - N_1 + 1}{2}\right]$, *where h is the dual mirror filter.*

It follows from the properties of AFs [2] that their Fourier spectra do not vanish at $\omega = \pi$. Therefore, AFs cannot be used as $h(\omega)$ (otherwise, one of the assertions of Theorem 1 is false). To satisfy the conditions of Theorem 1, we write $h(\omega)$ in the form $h(\omega) = \sqrt{2}\left(\cos\left(\frac{\omega}{2}\right)\right)^p \cdot \tilde{\theta}(\omega)$. Here, $\tilde{\theta}(\omega)$ is the Fourier-spectrum of the AF. On the time domain, such a function (e.g., the AF up(t)) can be represented by using the Laplace transform. Some basic properties of the AF up(t), which refer to problems of operational calculus, are given in [1, 2]. In this case, $h(\omega)$ has p zero moments for $\omega = \pi$. Such an approach makes it possible to use AFs in constructing new classes of wavelet functions. Since the scaling function $\varphi(x)$ must be orthonormal, $h(\omega)$ must have some special properties,

[a] *Institute of Radio Engineering and Electronics, Russian Academy of Sciences, ul. Mokhovaya 11, korp. 7, Moscow, 125009 Russia*
e-mail: olegk@lianet.ru
[b] *Central Design Bureau of Unique Instrumentation, Russian Academy of Sciences, ul. Butlerova 15, Moscow, 117342 Russia*

namely, $|h(\omega)|^2 + |h(\omega+\pi)|^2 = 1$. According to [1, 3], if a 2π-periodic function $h(\omega)$ has these special properties and the product $\frac{1}{(2\pi)^{1/2}}\prod_{j=1}^{\infty} h(2^{-j}\omega)$ converges, then its limit $\varphi(\omega)$ belongs to $L^2(R)$, and $\|\varphi\|_{L^2} \le 1$. In this case, the scaling function is defined by $\varphi(\omega) = \prod_{i=1}^{\infty}\frac{1}{\sqrt{2}}h(2^{-i}\omega)$. Having determined the scaling function $\varphi(x)$, we can find the wavelet generating function ψ. For example, ψ can be defined as

$$\psi(\omega) = \frac{1}{\sqrt{2}}e^{-i\frac{\omega}{2}}\overline{h\left(\frac{\omega}{2}+\pi\right)}\varphi\left(\frac{\omega}{2}\right)$$

or, equivalently,

$$\psi(x) = \sqrt{2}\sum_{n=-\infty}^{+\infty}(-1)^{n-1}h(-n-1)\varphi(t-n).$$

Then, $\psi(x)$ and $\varphi(x)$ are compactly supported functions from $L^2(R)$ satisfying the equations

$$\varphi(x) = \sqrt{2}\sum_{n=-\infty}^{+\infty}h_n\varphi(2x-n),$$

$$\psi(x) = \sqrt{2}\sum_{n=-\infty}^{\infty}(-1)^n h_{-n+1}\varphi(2x-n),$$

where the h_n are given in the form

$$h(\omega) = \frac{1}{\sqrt{2}}\sum_{n=-\infty}^{+\infty}h_n e^{-in\omega}.$$

The algorithm for constructing WA-systems of Kravchenko–Rvachev functions is as follows.

(i) Transforming the AF spectrum by the formula $h(\omega) = \sqrt{2}\left(\cos\left(\frac{\omega}{2}\right)\right)^p\tilde{\theta}(\omega)$.

(ii) Constructing a scaling function of the form $\varphi(\omega) = \prod_{i=1}^{\infty}\frac{1}{\sqrt{2}}h(2^{-i}\omega)$ with finitely many multipliers. The number of multipliers must be larger than 2^p, where p is the number of nonzero moments.

(iii) Constructing a wavelet generating function on the basis of the AF in the form

$$\psi(\omega) = \frac{1}{\sqrt{2}}e^{-i\frac{\omega}{2}}\overline{h\left(\frac{\omega}{2}+\pi\right)}\varphi\left(\frac{\omega}{2}\right),$$

or, equivalently, $\psi(x) = \sqrt{2}\sum_{n=-\infty}^{+\infty}(-1)^{n-1}h(-n-1)\varphi(t-n)$, where $\bar{h}(\omega)$ is conjugate to $h(\omega)$. The resulting wavelet generating function must satisfy the following conditions.

A. The zero mean condition: $\int_{-\infty}^{\infty}\psi(x)dx = 0$. A numerical experiment shows that it does hold.

B. The stability condition: $A \le \sum_{j=-\infty}^{\infty}|\hat{\psi}(2^{-j}\omega)|^2 \le B$. This condition also holds, because the function $\psi(x)$ is obtained by transforming the AF spectrum, which is bounded from above by $\max(\widehat{\mathrm{up}}(\omega)) = 1$ and rapidly decreases to zero. We do not multiply the AF by any quantity removing the upper bound for the spectrum amplitude. A numerical experiment confirms that this condition is satisfied.

C. The orthogonality condition: for $\psi_{j,k}(x) = 2^{j/2}\psi(2^jx-k)$ and $\psi_{l,m}(x) = 2^{l/2}\psi(2^lx-m)$, where $j, k, l, m \in Z$, $\langle\psi_{j,k}, \psi_{l,m}\rangle = \delta_{j,l}\delta_{k,m}$. A numerical experiment shows that the Kravchenko–Rvachev wavelet functions constructed above are orthogonal.

CONSTRUCTION OF COMPLEX KRAVCHENKO–RVACHEV WAVELETS

Construction procedure. The complex Kravchenko–Rvachev wavelets [1] are based on the following convolution of an AF and the multiplier $\cos^p\frac{\omega}{2}$, where $p \in \mathbb{N}$:

$$\hat{h}^p_\theta(\omega) = \left(\cos\frac{\omega}{2}\right)^p\hat{\theta}(\omega), \quad (2)$$

where $\hat{\theta}(\omega)$ is the spectrum of the AF. The spectrum of the family of complex wavelets (without normalization) has the form

$$\hat{\hat{\psi}}^p_\theta = \hat{h}^p_\theta(\omega-\pi). \quad (3)$$

To find an analytical representation for the wavelets, we must determine the inverse Fourier transform for expression (3). For this purpose, consider the inverse Fourier transform of $\cos^p\frac{\omega}{2}$. We use the following representation for odd p:

$$\cos^p\frac{\omega}{2} = \frac{1}{2^{p-1}}\left[\cos\left(p\frac{\omega}{2}\right) + C_p^1\cos\left((p-2)\frac{\omega}{2}\right) + C_p^2\cos\left((p-4)\frac{\omega}{2}\right) + \ldots + C_p^{\frac{p-1}{2}}\cos\left(\frac{\omega}{2}\right)\right], \quad (4)$$

where $C_p^k = \frac{p!}{k!(p-k)!}$. For even p, we have

$$\cos^p\left(\frac{\omega}{2}\right) = \frac{1}{2^{p-1}}\left[\cos\left(p\frac{\omega}{2}\right) + C_p^1\cos\left((p-2)\frac{\omega}{2}\right) + C_p^2\cos\left((p-4)\frac{\omega}{2}\right) + \ldots + C_p^{\frac{p-2}{2}}\cos\left(\frac{\omega}{2}\right)\right] + \frac{1}{2^p}C_p^{\frac{p}{2}}. \quad (5)$$

It is known that

$$F^{-1}\{\cos(a\omega)\} = \frac{\delta(t+a)+\delta(t-a)}{2}. \quad (6)$$

According to (4)–(6), for odd p, we have

$$F^{-1}\left\{\cos^p\left(\frac{\omega}{2}\right)\right\} = \frac{1}{2^p}\sum_{k=0}^{(p-2)/2} C_p^k\left[\delta\left(t+\frac{p-2k}{2}\right) + \delta\left(t-\frac{p-2k}{2}\right)\right], \quad (7)$$

and for even p, we have

$$F^{-1}\left\{\cos^p\left(\frac{\omega}{2}\right)\right\} = \frac{1}{2^p}\left[\sum_{k=0}^{(p-2)/2} C_p^k\left[\delta\left(t+\frac{p-2k}{2}\right) + \delta\left(t-\frac{p-2k}{2}\right)\right] + C_p^{\frac{p}{2}}\delta(t)\right]. \quad (8)$$

Let us find the inverse Fourier transform for (2) by using (7) and (8). For even p, we obtain

$$\bar{h}_\theta^p(t) = \frac{1}{2^p}\left[\sum_{k=0}^{(p-2)/2} C_p^k\left(\theta\left(t+\frac{p-2k}{2}\right) + \theta\left(t-\frac{p-2k}{2}\right)\right) + C_p^{\frac{p}{2}}\theta(t)\right]. \quad (9)$$

Similarly, for odd p, we have

$$\tilde{h}_\theta^p(t) = \frac{1}{2^p}\sum_{k=0}^{(p-1)/2} C_p^k\left(\theta\left(t+\frac{p-2k}{2}\right) + \theta\left(t-\frac{p-2k}{2}\right)\right). \quad (10)$$

It follows from (3) that

$$\tilde{\psi}_\theta^p(t) = e^{i\pi t}h_\theta^p(t), \quad (11)$$

$$h_\theta^p(t) = \begin{cases}\bar{h}_\theta^p(t), & p \text{ is even},\\ \tilde{h}_\theta^p(t), & p \text{ is odd}.\end{cases} \quad (12)$$

Therefore, the complex Kravchenko–Rvachev wavelets are determined up to a normalizing multiplier by expressions (9)–(12). However, in practice, wavelets must have unit norm in the space L_2. Let us determine the norm of function (11). For this purpose, we calculate $\|h_\theta^p(t)\|$ with taking into account the fact that the shift of a function does not affect its norm in the space L_2. After transformations, we obtain the following expression for even p:

$$\|\bar{h}_\theta^p(t)\| = \frac{1}{2^p}\Bigg(2\sum_{k=0}^{(p-2)/2}(C_p^k)^2\|\theta(t)\|^2 + (C_p^{p/2})^2\|\theta(t)\|^2 + 2\int_{-\infty}^{\infty}\Bigg[\sum_{k=0}^{(p-2)/2}\left(C_p^kC_p^{p/2}\theta\left(t+\frac{p-2k}{2}\right)\theta(t)\right) + \sum_{k=0}^{(p-2)/2}\left(C_p^kC_p^{p/2}\theta\left(t-\frac{p-2k}{2}\right)\theta(t)\right) + \sum_{i=0}^{(p-2)/2}\sum_{j=0}^{(p-2)/2}C_p^iC_p^j\theta\left(t+\frac{p-2i}{2}\right)\theta\left(t-\frac{p-2j}{2}\right)\Bigg]^2 dt\Bigg)^{1/2}. \quad (13)$$

Similarly, for odd p, we have

$$\|\tilde{h}_\theta^p(t)\| = \frac{1}{2^p}\Bigg(2\sum_{k=0}^{(p-1)/2}(C_p^k)^2\|\theta(t)\|^2 + 2\sum_{i=0}^{(p-1)/2}\sum_{j=0}^{(p-1)/2}C_p^iC_p^j\theta\left(t+\frac{p-2i}{2}\right)\theta\left(t-\frac{p-2j}{2}\right)\Bigg]^2 dt\Bigg)^{1/2}. \quad (14)$$

A shift of a function in the frequency domain does not affect its L_2-norm either; therefore,

$$\|\tilde{\psi}_\theta^p(t)\| = \|e^{i\pi t}h_\theta^p(t)\| = \|h_\theta^p(t)\|. \quad (15)$$

The expression for the above family of Kravchenko–Rvachev wavelets with unit norm has the form

$$\psi_\theta^p(t) = \frac{1}{\|h_\theta^p(t)\|}e^{i\pi t}h_\theta^p(t), \quad (16)$$

where $h_\theta^p(t)$ is determined by (12).

The Fourier transform of wavelets (16) is

$$\hat{\psi}_\theta^p(\omega) = \frac{\sqrt{2\pi}}{\|\hat{h}_\theta^p(\omega)\|}\hat{h}_\theta^p(\omega-\pi), \quad p \in \mathbb{N}, \quad (17)$$

where $\hat{h}_\theta^p(\omega)$ is determined by (2).

Table 1. The frequency spreads and supports of the complex Kravchenko–Rvachev wavelets $\psi_\theta^p(t)$ based on the AFs $\mathrm{up}_m(t)$

p	$\mathrm{up}_1(t)$		$\mathrm{up}_2(t)$		$\mathrm{up}_3(t)$		$\mathrm{up}_4(t)$		$\mathrm{up}_5(t)$	
	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp
1	1.78	[–1.5, 1.5]	1.56	[–1.5, 1.5]	1.33	[–1.5, 1.5]	1.19	[–1.5, 1.5]	1.17	[–1.5, 1.5]
2	1.15	[–2, 2]	0.97	[–2, 2]	0.83	[–2, 2]	0.74	[–2, 2]	0.74	[–2, 2]
3	0.85	[–2.5, 2.5]	0.71	[–2.5, 2.5]	0.60	[–2.5, 2.5]	0.54	[–2.5, 2.5]	0.54	[–2.5, 2.5]
4	0.67	[–3, 3]	0.56	[–3, 3]	0.47	[–3, 3]	0.43	[–3, 3]	0.42	[–3, 3]
5	0.56	[–3.5, 3.5]	0.46	[–3.5, 3.5]	0.39	[–3.5, 3.5]	0.35	[–3.5, 3.5]	0.35	[–3.5, 3.5]

Graphs of normalized Kravchenko–Rvachev complex wavelets and their Fourier transforms are shown in Figs. 1–3.

Substantiation. Let us show that functions (16) obtained above are wavelets. For this purpose, we must prove that they satisfy the conditions [4]

$$\psi_\theta^p \in L_2, \quad \|\psi_\theta^p\| = 1; \tag{18}$$

$$2\pi\int_{-\infty}^{\infty}\frac{|\hat{\psi}_\theta^p(a)|^2}{|a|}da = C_\psi < \infty. \tag{19}$$

If (18) holds, then condition (19) is equivalent to

$$\int_{-\infty}^{\infty}|t||\psi_\theta^p(t)|dt < \infty; \tag{20}$$

$$\int_{-\infty}^{\infty}\psi_\theta^p(t)dt = 0, \text{ or, equivalenty} \quad \hat{\psi}_\theta^p(0) = 0. \tag{21}$$

It follows from (9)–(12) that the wavelets under consideration are products of linear combinations of finitely many shifted AFs and the bounded continuous function $e^{i\pi t}$. Therefore, they inherit the following properties of the AFs from which they are obtained: continuity, compact support, and boundedness. These properties ensure the fulfillment of conditions (18) and (20) (that they have unit norm is ensured by the above normalization of function (11)). Thus, it remains to show that functions (16) satisfy the zero mean condition (21). This follows from the vanishing of the Fourier transform (17) at zero:

$$\hat{\psi}_\theta^p(0) = \frac{1}{\|h_\theta^p(t)\|}\hat{h}(-\pi) = \frac{1}{\|h_\theta^p(t)\|}\left(\cos\frac{\pi}{2}\right)^p\hat{\theta}(-\pi) = 0. \tag{22}$$

PROPERTIES OF COMPLEX KRAVCHENKO–RVACHEV WAVELETS

In this section, we consider properties of the new family of Kravchenko–Rvachev wavelets.

Zero mean. As shown above (see (21), (22)), the complex Kravchenko–Rvachev wavelets satisfy the zero mean condition

$$\int_{-\infty}^{+\infty}\psi_\theta^p(t)dt = 0.$$

Smoothness. The complex Kravchenko–Rvachev wavelets have the same degree of smoothness as the AFs from which they are obtained. Indeed, wavelets (16) are (up to multiplication by a constant) products of $e^{i\pi t}$, which is an infinitely differentiable function, and the function $h_\theta^p(t)$, which is the sum of weighted shifts by the argument of the AF $\theta(t)$. Therefore, for infinitely differentiable AFs, complex wavelets (16) are infinitely differentiable as well.

The size of the support. The complex Kravchenko–Rvachev wavelets are compactly supported, because so are AFs. The support of wavelets (16) coincides with that of the function $h_\theta^p(t)$, which, in turn, depends on the support of the AFs. The form of the dependence is easy to determine by using (9)–(11). Suppose that the AF $\theta(t)$ has support $\mathrm{supp}\,\theta(t) = [a, b]$ and the support of its complex wavelet is

$$\mathrm{supp}\,\psi_\theta^p(t) = \left[a - \frac{p}{2}, b + \frac{p}{2}\right]. \tag{23}$$

It follows from (23) that the size of the support of a wavelet increases with its order p. The sizes of the supports of some complex Kravchenko–Rvachev wavelets are given in Tables 1–3.

Frequency resolution. The frequency resolution of a wavelet $\psi(t)$ is defined as the reciprocal of its frequency spread [4]:

$$\sigma_\omega^2 = \frac{1}{2\pi}\int_0^{\infty}(\omega - \eta)^2|\hat{\psi}(\omega)|^2 d\omega, \tag{24}$$

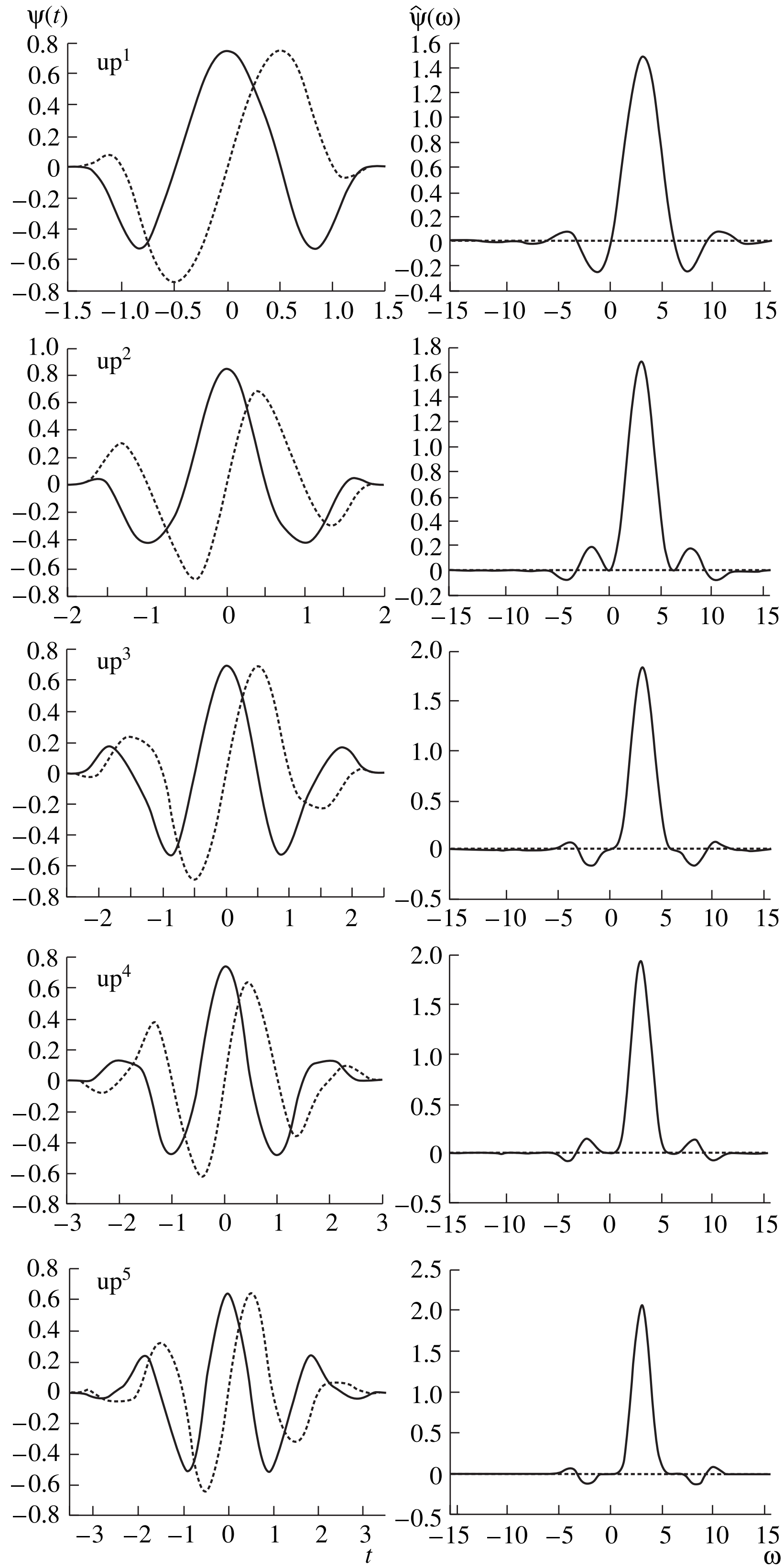

Fig. 1. Left: the graphs of complex Kravchenko–Rvachev wavelets $\psi(t)$ based on the AFs up(t) of orders 1–5 (the dashed lines show their imaginary parts); right: the corresponding Fourier transforms $\hat{\psi}(\omega)$ of the wavelets.

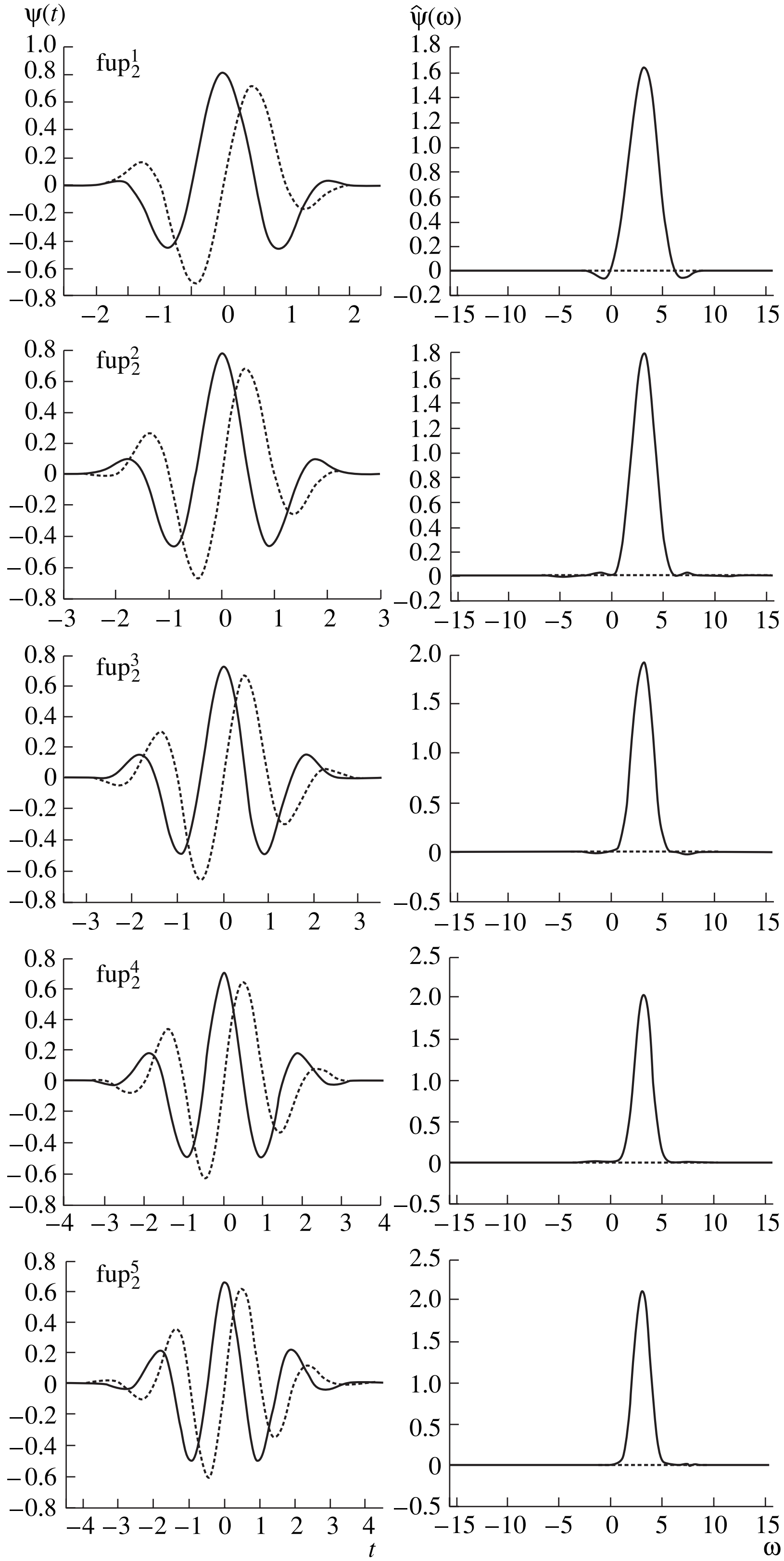

Fig. 2. Left: the graphs of complex Kravchenko–Rvachev wavelets $\psi(t)$ based on the AFs $\mathrm{fup}^2(t)$ of orders 1–5 (the dashed lines show their imaginary parts); right: the corresponding Fourier transforms $\hat{\psi}(\omega)$ of the wavelets.

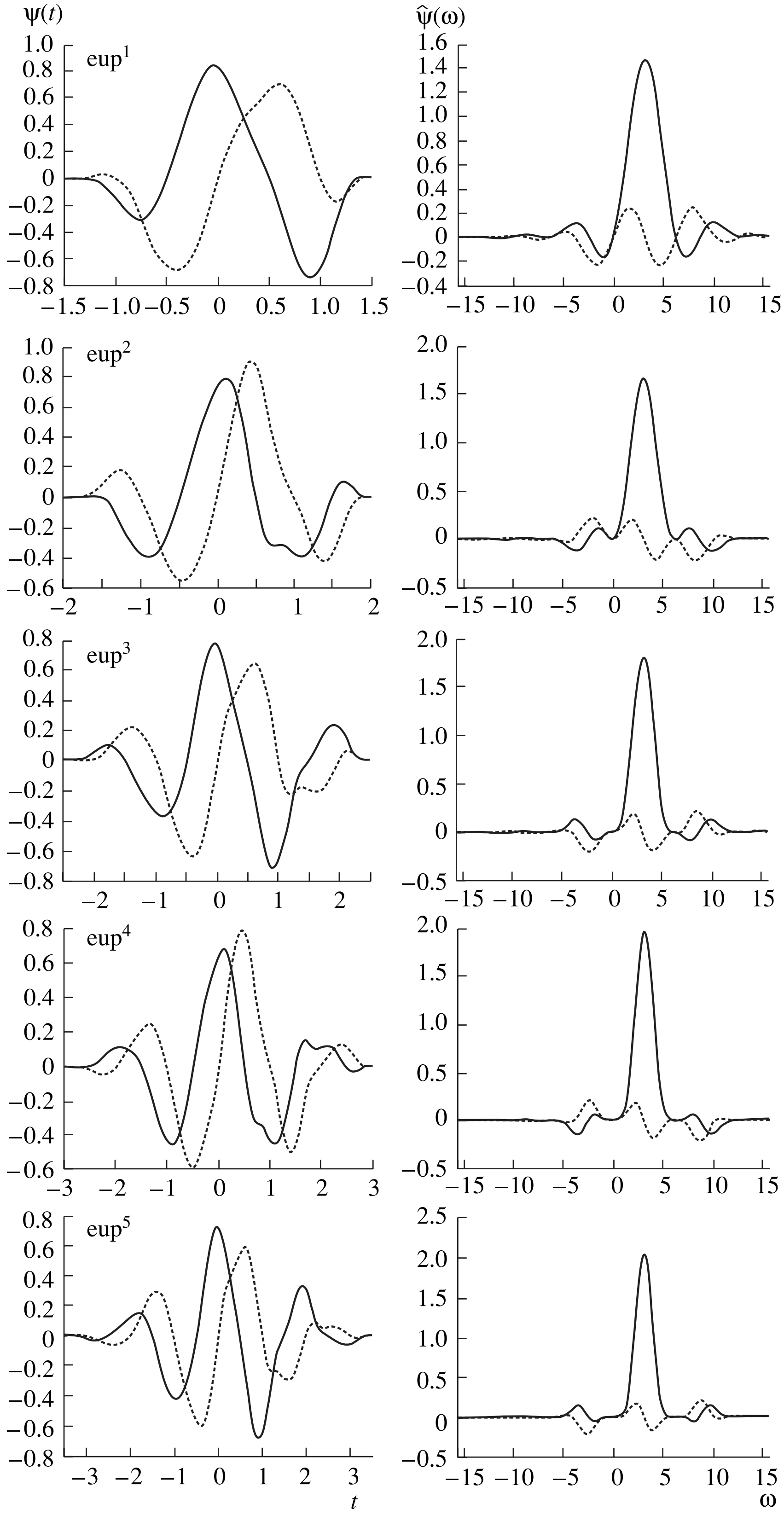

Fig. 3. Left: the graphs of complex Kravchenko–Rvachev wavelets $\psi(t)$ based on the AFs $\mathrm{eup}(t)$ of orders 1–5 (the dashed lines show their imaginary parts); right: the corresponding Fourier transforms $\hat{\psi}(\omega)$ of the wavelets.

Table 2. The frequency spreads and supports of the complex Kravchenko–Rvachev wavelets $\psi_\theta^p(t)$ based on the AFs $\mathrm{fup}_n(t)$

p	$\mathrm{fup}_1(t)$		$\mathrm{fup}_2(t)$		$\mathrm{fup}_3(t)$		$\mathrm{fup}_4(t)$		$\mathrm{fup}_5(t)$	
	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp
1	1.00	[–2, 2]	0.83	[–2.5, 2.5]	0.73	[–3, 3]	0.65	[–3.5, 3.5]	0.59	[–4, 4]
2	0.65	[–2.5, 2.5]	0.57	[–3, 3]	0.53	[–3.5, 3.5]	0.49	[–4, 4]	0.45	[–4.5, 4.5]
3	0.49	[–3, 3]	0.44	[–3.5, 3.5]	0.41	[–4, 4]	0.39	[–4.5, 4.5]	0.37	[–5, 5]
4	0.39	[–3.5, 3.5]	0.36	[–4, 4]	0.34	[–4.5, 4.5]	0.32	[–5, 5]	0.31	[–5.5, 5.5]
5	0.33	[–4, 4]	0.31	[–4.5, 4.5]	0.29	[–5, 5]	0.28	[–5.5, 5.5]	0.27	[–6, 6]

Table 3. The frequency spreads and supports of the complex Kravchenko–Rvachev wavelets $\psi_\theta^p(t)$ based on the AFs $\mathrm{eup}_a(t)$

p	$\mathrm{eup}_1(t)$		$\mathrm{eup}_2(t)$		$\mathrm{eup}_3(t)$		$\mathrm{eup}_4(t)$		$\mathrm{eup}_5(t)$	
	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp	σ_ω^2	supp
1	2.11	[–1.5, 1.5]	2.62	[–1.5, 1.5]	3.15	[–1.5, 1.5]	3.66	[–1.5, 1.5]	4.14	[–1.5, 1.5]
2	1.47	[–2, 2]	1.98	[–2, 2]	2.50	[–2, 2]	3.02	[–2, 2]	3.51	[–2, 2]
3	1.16	[–2.5, 2.5]	1.67	[–2.5, 2.5]	2.20	[–2.5, 2.5]	2.71	[–2.5, 2.5]	3.20	[–2.5, 2.5]
4	0.99	[–3, 3]	1.49	[–3, 3]	2.01	[–3, 3]	2.53	[–3, 3]	3.02	[–3, 3]
5	0.87	[–3.5, 3.5]	1.37	[–3.5, 3.5]	1.89	[–3.5, 3.5]	2.41	[–3.5, 3.5]	2.90	[–3.5, 3.5]

where $\eta = \frac{1}{2\pi}\int_0^{+\infty}\omega|\hat{\psi}(\omega)|^2 d\omega$ is the frequency center of the function $\hat{\psi}(\omega)$. For all wavelets considered in this paper, the frequency center is $\eta = \pi$. This value is obtained as a result of the frequency shift (17) of the corresponding AFs, whose frequency centers are at zero. The frequency spreads of some complex Kravchenko–Rvachev wavelets are given in Tables 1–3. These tables show that the frequency resolution of a wavelet increases with its order p. The results obtained agree with the Heisenberg uncertainty principle.

The parity of the real and imaginary parts. For complex functions, it makes sense to consider the parities of their real and imaginary parts separately. The construction of complex Kravchenko–Rvachev wavelets ensures that the real part of an even AF is an even function, and its imaginary part is an odd function:

$$\begin{aligned} \mathrm{Re}(\psi_\theta^p(-t)) &= \mathrm{Re}(\psi_\theta^p(t)), \\ \mathrm{Im}(\psi_\theta^p(-t)) &= -\mathrm{Im}(\psi_\theta^p(t)) \quad \text{as} \quad \theta(t) = \theta(-t). \end{aligned} \tag{25}$$

CONCLUSIONS

A new class of WA-systems of Kravchenko–Rvachev functions and a family of complex wavelets are obtained and substantiated. Their basic properties are studied. It is shown that complex wavelets inherit these properties from the AFs from which they are constructed. Note that, in main formula (2), the spectrum of an arbitrary compactly supported function-window can be used for $\hat{\theta}(\omega)$, and the shift of the spectrum in relation (17) can be any odd number multiplied by π. Large shifts of the spectrum may be useful in applying analytic wavelet-transforms.

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