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MATHEMATICAL PHYSICS

Study of the Structure of Solutions to Nonlinear Wave Equations Based on Continuous Wavelet Analysis

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Nonlinear waves and oscillations occupy a special place in modern physics and radio physics [1–7]. However, physical phenomena associated with nonlinear processes have not been thoroughly investigated thus far [7]. Mathematical methods based on wavelet analysis [8–14] can be promising as applied to the detection of new features of nonlinear processes, waves, and oscillations. In this paper, a new approach based on the continuous wavelet transform is proposed to study the structure of solutions to standard nonlinear equations and the results are compared with those based on the window Fourier transform.

BASIC NONLINEAR WAVE PROCESSES

We analyze shock waves governed by the Burgers equation and the modified Burgers equation with cubic nonlinearity, classical and dissipative solitons, soliton-like solutions to the modified Korteweg–de Vries (KdV) equation and the sine-Gordon equation, kinks, and antikinks. Primary attention is given to the analysis of sawtooth waves, centaur solutions, cnoidal waves, and envelope solitons.

Sawtooth waves. Waves with a sawtooth profile originate when nonlinear steepening dominates over dissipation. These waves are not steady. Like usual steady shock waves, they are described by the Burgers equation. The solution of the Burgers equation was first found by R.V. Khokhlov and is well known [3].

Centaur Solution. When nonlinearity, dissipation, and dispersion are taken into account, we have to solve the Burgers–Korteweg–de Vries (BKdV) equation [1–7]. For steady waves with $\nu(-\infty) = \nu_1$ and $\nu(+\infty) = \nu'(\pm\infty) = \nu''(\pm\infty) = 0$, the solution to the BKdV equation, the so-called centaur solution, has the form shown in Fig. 1a. It describes a shock wave propagating in a dispersive medium.

Cnoidal waves. In addition to soliton solutions, the KdV equation can have periodic solutions describing nonlinear (cnoidal) waves [1–7].

Envelope soliton. It is the solution to the nonlinear Schrödinger equation [1–7].

CONTINUOUS WAVELET TRANSFORM

It is well known that the continuous wavelet transform of a signal $s(t)$ is based on a mother wavelet function from which a wavelet basis is constructed by using translations (b is a shift parameter) and dilations (a is a scaling parameter) [8–11, 13]. In this paper, the wavelets are denoted as in Matlab [11]. Specifically, the Daubechies wavelets of order 1, 2, …, 12 are denoted by db1, db2, …, db12, respectively; the Symlet wavelets of order 2, 3, …, 10, by sym2, sym3, …, sym10; the coiflets of order 1, 2, …, 5, by coif1, coif2, …, coif5; the Gaussian wavelets of order 1, 2, …, 8, by gaus1, gaus2, …, gaus8; the Morlet wavelet, by morl; the Haar wavelet, by haar; the MHAT wavelet, by mexh; and the Meyer wavelet, by meyr. In addition to the analysis of the moduli of wavelet spectra $|Wf(a, b)|$, it is also useful to examine skeletons and scalograms [9–11]. It is convenient to use the continuous wavelet transform of a signal together with its spectrogram based on the window Fourier transform and also with its skeleton of the spectrogram and the one-dimensional Fourier spectrum. The parameters a and b of the continuous wavelet transform are related to the parameters ω and τ of the window Fourier transform by the simple formulas $b = \tau$ and $a = kT = \frac{2\pi k}{\omega}$, where k is a proportionality constant ($k > 0$) depending on the chosen function $\psi(t)$. For example, for the MHAT wavelet (mexh) and the fourth-order Daubechies wavelet (db4), $k \approx 0.250$ and $k \approx 0.714$, respectively. For convenience of comparing spectrograms with wavelet spectra parameters, a and ω

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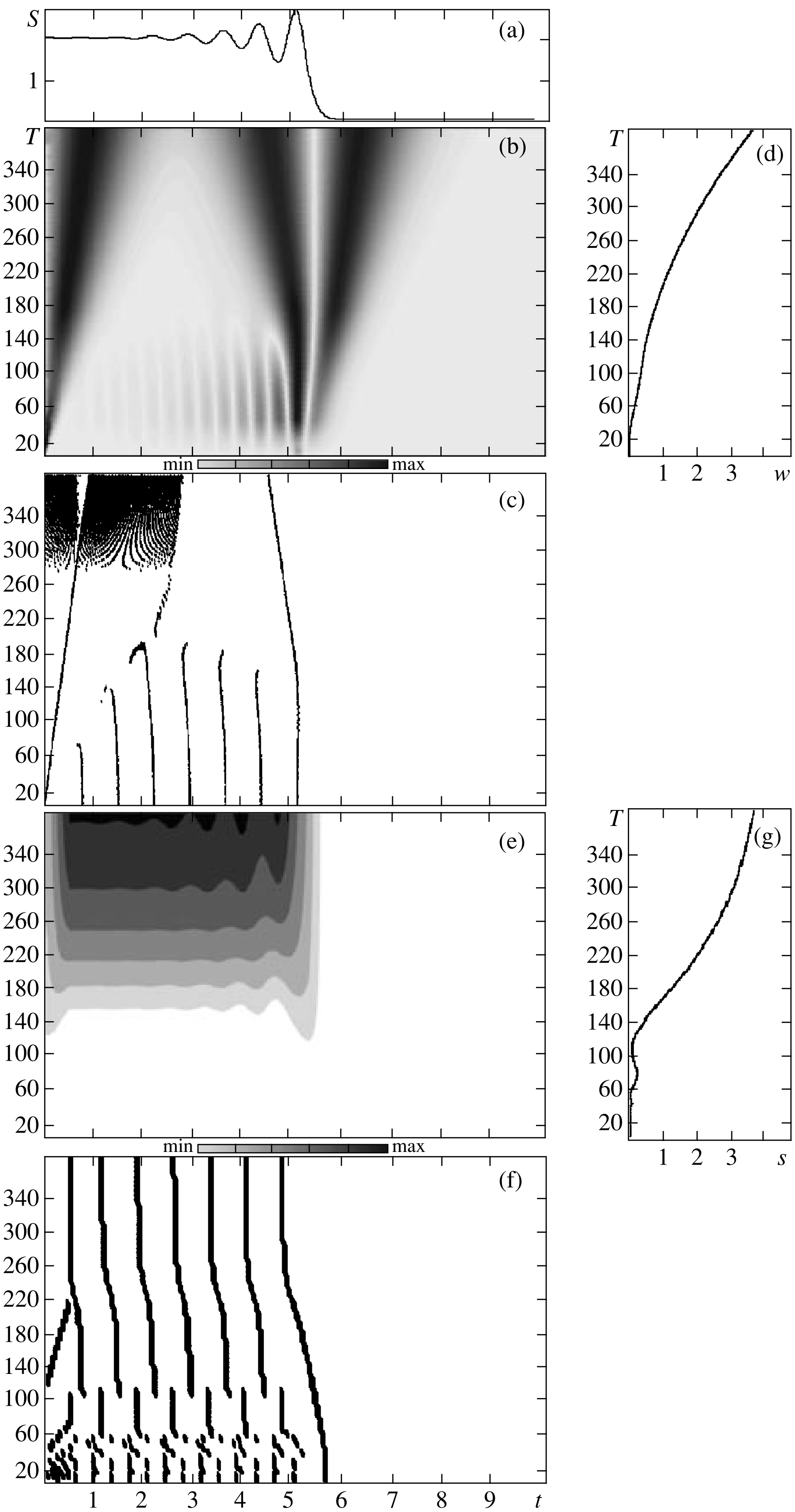

Fig. 1. Analysis of the shock wave (centaur solution) in a dispersive medium: (a) the shock wave in the time domain, (b) the wavelet spectrogram obtained using the mexh wavelet, (c) scalogram, (d) skeleton, (e) the spectrogram of the window Fourier transform, (f) the one-dimensional Fourier spectrum, and (g) the skeleton of the spectrogram.

were replaced with the corresponding period T, and the parameters b and τ were replaced with t.

WAVELET ANALYSIS OF SOLUTIONS TO THE NONLINEAR EQUATIONS

The results are represented in a specially developed format (Fig. 1). The numerical analysis was performed for various real wavelets and revealed the following features.

The behavior of the characteristics of the envelope soliton is typical of wave packets. The wavelet spectra and the skeletons are symmetric for the symmetric wavelet mexh and have a certain asymmetry for asymmetric wavelets, in particular, for the fractal wavelet db4. For the centaur solution, the skeleton exhibits three diverging lines characteristic of solitons and, to the left of them, equidistant lines corresponding to the oscillating tail, which are somewhat smeared by the edge effect. The wavelet spectrum and the skeleton of sawtooth waves are hardly distinguishable from those of shock waves. The fact is that the real wavelet transform of a constant or a linear function is zero. For this reason, the difference between the nearly linear segments of a sawtooth wave and the constant segments of a shock wave is indistinguishable. The former wave can be distinguished from the latter in the spectrogram of the window Fourier transform and in its skeleton. The wavelet spectrum of a cnoidal wave resembles that of a series of skeletons. The same analysis can be performed for the spectrogram of its window Fourier transform and for the skeleton of the spectrogram. A distinctive feature is that the skeleton exhibits vertical lines corresponding to the troughs of the cnoidal wave. In the wavelet analysis of any signal, it is important to choose a proper wavelet basis. The character of the resulting wavelet spectrum depends on both the underlying signal and the wavelet used.

Since wavelets can be treated as frequency filters, it was proposed in [13, 14] to characterize them by the following numerical characteristics: the broadbandness parameter μ; the relative position of the FSP's maximum γ_1; the relative position of the FSP's first zero γ_2; the FSP width at 3 dB (γ_3) and at 6 dB (γ_4); the information loss γ_5 (in dB) after discarding from the FSP the components with the frequencies $f \notin [0, \nu]$, where ν is the first zero of the FSP to the right of its principal maximum f_m; the information loss γ_6 (in dB) after discarding from the FSP the components with the frequencies $f \notin [f_{\min}, f_{\max}]$, where $f_{\min}$ and $f_{\max}$ are determined by decaying the FSP by e times with respect to its principal maximum; the coherent gain γ_7; the equivalent noise band γ_8; and the maximum level of the side lobes γ_9 (in dB). These characteristics can be conveniently used to describe the nonlinear functions under study. The computed values of μ, γ_1, γ_2, …, γ_9 for various models of nonlinear waves and oscillations are listed in Table 1, where $s_1(t)$ is a shock wave described by the BKdV equation, $s_2(t)$ is an envelope soliton, $s_3(t)$ is a sawtooth wave, and $s_4(t)$ is a cnoidal wave.

Table 1. Parameters of the models of nonlinear waves and oscillations

Parameters	Models of nonlinear waves and oscillations			
	$s_1(t)$	$s_2(t)$	$s_3(t)$	$s_4(t)$
μ	2.00	0.56	0.92	2.00
γ_1	0.00	0.98	0.69	0.00
γ_2	22.54	2.21	9.48	5.51
γ_3	1.04	0.30	0.15	1.01
γ_4	1.68	0.45	0.25	1.10
$-\gamma_5$, dB	5.08	33.69	16.15	21.30
$-\gamma_6$, dB	2.50	7.17	2.79	7.86
γ_7	0.40	0.06	0.54	0.25
γ_8	1.87	8.30	1.32	2.70
$-\gamma_9$, dB	7.15	97.31	3.03	0.97

It was established that some of the physical parameters are identical for all the nonlinear waves and oscillations ($\mu = 2$, $\gamma_1 = 0$, $\gamma_3 \approx 1.0$–1.2, $\gamma_4 \approx 1.6$–1.7) except for the envelope soliton ($\mu \approx 0.6$, $\gamma_1 \approx 1$, $\gamma_3 \approx 0.3$, $\gamma_4 \approx 0.5$), the sawtooth wave ($\mu \approx 0.9$, $\gamma_1 \approx 0.7$, $\gamma_4 \approx 0.3$), and the cnoidal wave ($\gamma_4 \approx 1.1$).

Other parameters differ by several times for shock waves and solitons (in brackets): $\gamma_5 \approx 6$–8 ($\gamma_5 \approx 25$–26), $\gamma_6 \approx 2$–3 ($\gamma_6 \approx 6$–7), and $\gamma_8 \approx 1.8$–2.0 ($\gamma_8 \approx 10$–17). Finally, there are parameters that differ considerably: $\gamma_2 \approx 80$–170 ($\gamma_2 \approx 6$–8), $\gamma_7 \approx 0.5$ ($\gamma_7 \approx 0.04$–0.06), and $\gamma_9 \approx 6$–8 ($\gamma_9 \approx 120$) for shock waves and solitons, respectively. For the envelope soliton, the parameters in the second and third groups are close to those for solitons. All the parameters of the centaur solution, except for γ_2, are close to those of shock waves, and $\gamma_2 \approx 23$ occupies an intermediate position between the shock waves and solitons, which apparently justifies the name of this nonlinear wave. For the sawtooth wave, $\gamma_6 \approx 2.8$ and $\gamma_7 \approx 0.5$ are close to these parameters for shock waves, and $\gamma_2 \approx 9.5$ is close to the corresponding value for solitons. At the same time, γ_5, γ_8, and γ_9 for the sawtooth wave, shock waves, and solitons differ.

For the cnoidal wave, γ_2, γ_5, and γ_6 are close to the corresponding parameters for solitons ($\gamma_2 \approx 5.5$, $\gamma_5 \approx 21.3$, $\gamma_6 \approx 7.9$). The remaining parameters differ. The fact is that the cnoidal wave is an intermediate structure between a harmonic wave and a soliton.

To evaluate the choice of the wavelet basis and to analyze a particular signal, it can be helpful to use a cost functional of the form

$$J(f_1, f_2) = \sum_{i=1}^{10}\left(\frac{\gamma_i(f_1) - \gamma_i(f_2)}{\gamma_i(f_2)}\right)^2, \quad \gamma_{10}(f) \equiv \mu(f),$$

Table 2. Cost functional $J(f_1, f_2)$ for the models of nonlinear waves and oscillations for the Haar, Daubechies, Symlet wavelets, coiflets, Gaussian, Morlet, and MHAT wavelets

Wavelets	Models of nonlinear waves and oscillations			
	$s_1(t)$	$s_2(t)$	$s_3(t)$	$s_4(t)$
haar (db1)	5.31	262	59.67	27.78
db2	2.74	16.70	33.22	48.40
db3	5.31	9.36	31.56	71.40
db4	9.38	7.75	33.92	101.1
db5	10.47	6.52	35.13	134.4
db6	14.44	5.76	39.92	170.5
db7	18.81	5.49	45.22	209.9
db8	19.84	5.23	48.40	252.0
db9	23.15	5.07	53.66	297.2
db10	27.28	4.68	59.76	346.0
sym2	2.74	16.66	33.20	48.40
sym3	5.31	9.39	31.82	71.39
sym4	10.40	6.48	36.87	101.5
sym5	11.88	5.36	38.47	134.9
sym6	17.82	4.43	47.46	171.9
sym7	22.51	4.18	53.42	211.4
coif1	18.42	11.98	53.89	81.14
coif2	37.81	10.61	75.81	161.6
coif3	60.99	11.14	104.2	255.3
coif4	81.11	11.80	137.6	362.7
coif5	102.7	12.63	175.2	484.4
gaus1	12.96	24.39	62.27	4.18
gaus2	14.53	12.83	39.95	4.41
gaus3	13.26	11.31	29.04	4.53
gaus4	12.88	8.61	23.36	4.74
gaus5	12.84	8.52	20.11	4.86
gaus6	13.15	6.87	18.32	5.05
gaus7	13.46	7.85	17.12	5.10
gaus8	13.96	6.50	16.70	5.25
morl	15.68	3.60	20.71	6.79
mexh	13.11	11.43	39.07	4.51

where f_1 denotes the wavelet function $\psi(t)$ and f_2 stands for the signal $s(t)$ to be analyzed. The optimal basis corresponds to the minimum value of this functional.

The numerical values of $J(f_1, f_2)$ for various models of nonlinear oscillations and wavelet bases are listed in Table 2. By applying the technique proposed, the optimal wavelets were found to be db2 and sym2 for all shock waves and the centaur solutions, sym6 for solitons, morl for the envelope soliton, gaus8 for the sawtooth wave, and gaus1 for the cnoidal wave.

Thus, it was shown that wavelet analysis is a promising tool for describing nonlinear wave processes. A new format of data representation combining the continuous wavelet transform with the window Fourier transform was developed. We calculated certain numerical characteristics of various nonlinear wave processes, which can be compared to each other. A cost functional was introduced that makes it possible to choose an optimal wavelet for each model of a signal. The numerical values of the functional were determined for physical models of nonlinear wave processes with the use of various wavelet bases.

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