ISSN 1064-5624, Doklady Mathematics, 2009, Vol. 80, No. 2, pp. 775–780. © Pleiades Publishing, Ltd., 2009.
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New System of Kravchenko Orthogonal Wavelets $\{\widetilde{\mathrm{up}(t)}\}$

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Received March 26, 2009

DOI: 10.1134/S1064562409050378

We propose and justify a new system of Kravchenko orthogonal wavelets $\{\widetilde{\mathrm{up}(t)}\}$ constructed on the basis of atomic functions (AFs) [1–6] invoking the ideas and results of [1–8].

1. The task is to construct AF-based wavelets such that the scaling function $\varphi(x)$ forms a collection of closed embedded subspaces $V_j \subset L^2(\mathbb{R})$ $(j \in \mathbb{Z})$ generating multiresolution analysis (MRA) [1–4, 6–8], which has the following properties:

(i) $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$.

(ii) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.

(iii) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1}$.

(iv) There exists a scaling function $\varphi(x) \in V_0$ such that its shifts form a Riesz basis for V_0.

Property (iv) is a fundamental one. According to [1, 6], the scaling function $\varphi(x)$ can be specified as a function whose Fourier transform $\hat{\varphi}(\omega)$ is a compactly supported even function. Below, wavelets are constructed using the atomic function $\mathrm{up}(t)$, since the basic elements of the construction algorithm are transferred to other AF families [6].

2. Let V_0 be the subspace of $L^2(\mathbb{R})$ generated by the shifts of $\varphi(x)$. For the functions $\varphi_n(x) = \varphi(x - n)$ to form a Riesz basis [6–8] for V_0, it is necessary that the following results hold true.

Theorem 1. *The system* $\{\varphi(x - n)\}_{n \in \mathbb{Z}}$ *obtained by the shifts of a function* $\varphi(x) \in L^2(\mathbb{R})$ *forms a Riesz basis of* $V_0 \subset L^2(\mathbb{R})$ *if and only if there exist positive constants A and B such that*

$$A \le \sum_{n \in \mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 \le B. \quad (1)$$

Moreover, the following theorem holds for the orthonormal functions $\varphi_n(x) = \varphi(x - n)$.

Theorem 2. *The functions* $\varphi_n(x) = \varphi(x - n)$ *form an orthonormal basis of* $V_0 \in L^2(\mathbb{R})$ *if and only if*

$$\sum_{n \in \mathbb{Z}} |\hat{\varphi}(\omega + 2\pi n)|^2 = 1. \quad (2)$$

First, we search for a function $\chi(\omega) = |\hat{\varphi}(\omega)|^2$ such that its shifts $\chi_n(\omega) = \chi_n(\omega + 2\pi n)$ form a partition of unity,

$$\sum_{n \in \mathbb{Z}} \chi(\omega + 2\pi n) = 1 \quad (3)$$

and it satisfies the conditions

$$\mathrm{supp}(\hat{f}(\omega)) = \left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right],$$

$$\hat{f}(\omega) = \begin{cases} 1 & \text{if } \omega \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right] \\ 0.5 & \text{if } \omega = \pi. \end{cases}$$

According to the properties of $\mathrm{up}(t)$ [3],

$$\sum_{n \in \mathbb{Z}} \mathrm{up}(t + n) = 1. \quad (4)$$

3. For the width of the flat top to be greater than or equal to a half of the function's support, we consider the partial sum (Fig. 1a)

$$\mathrm{up}_1^{\mathrm{sum}}(t) = \mathrm{up}(t+1) + \mathrm{up}(t) + \mathrm{up}(t-1). \quad (5)$$

Next, we pass from the support $\mathrm{supp}(\mathrm{up}_1^{\mathrm{sum}}(t)) = [-2, 2]$ to the support $\mathrm{supp}(\mathrm{up}_1^{\mathrm{sum}}(t))\left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$:

$$\mathrm{up}_2^{\mathrm{sum}}(t) = \mathrm{up}\left(\frac{3}{2\pi}t + 1\right)\ \mathrm{up}\left(\frac{3}{2\pi}t\right) + \mathrm{up}\left(\frac{3}{2\pi}t - 1\right). \quad (6)$$

We obtain a modified atomic function $\mathrm{up}_2^{\mathrm{sum}}(t)$ obeying all the conditions that have to be satisfied by $\chi(\omega)$ (Fig. 1b).

I make the formal substitution $t \to \omega$: $\chi(\omega) = \mathrm{up}_2^{\mathrm{sum}}(\omega)$. To determine the Fourier transform of the scaling function $\hat{\varphi}(\omega)$, it is necessary to extract the square root of $\chi(\omega)$:

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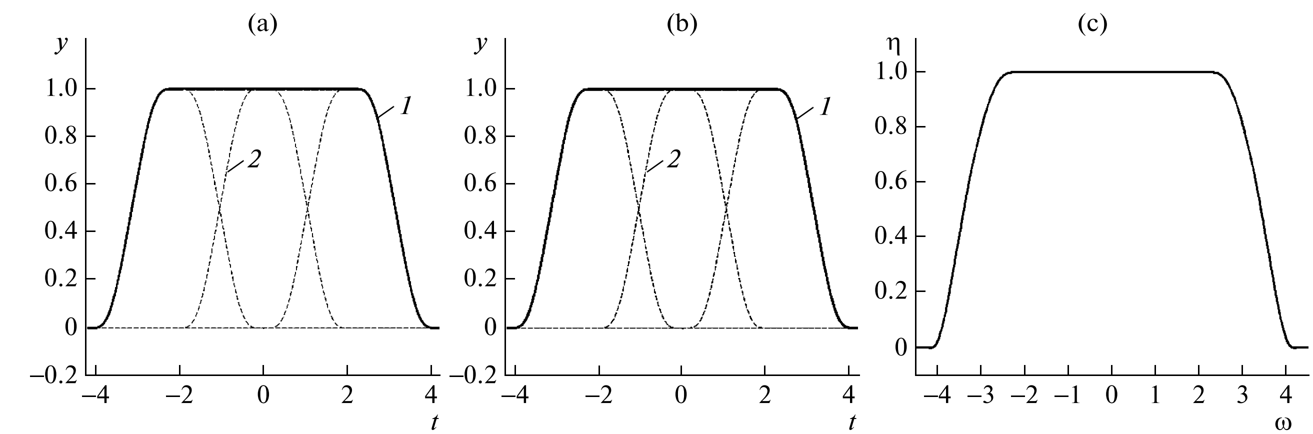

Fig. 1. (a) Plots of (*1*) $\mathrm{up}_1^{\mathrm{sum}}(t)$ and (*2*) $\mathrm{up}(t+k)$, $k = -1, 0, 1$. Plots of (b) $\mathrm{up}_2^{\mathrm{sum}}(\omega)$: (*1*) $\mathrm{up}_2^{\mathrm{sum}}(t)$ and (*2*) $\mathrm{up}\left(\frac{3}{2\pi}t + n\right)$, $k = -1$, 0, 1; and (c) $\hat{\varphi}(\omega)$ for the Kravchenko wavelet $\{\widetilde{\mathrm{up}}(\omega)\}$.

$$\hat{\varphi}(\omega) = \widetilde{\mathrm{up}}(\omega)$$
$$= \sqrt{\mathrm{up}\left(\frac{3}{2\pi}\omega + 1\right) + \mathrm{up}\left(\frac{3}{2\pi}\omega\right) + \mathrm{up}\left(\frac{3}{2\pi}\omega - 1\right)}. \qquad (7)$$

The plot of $\hat{\varphi}(\omega)$ for the new wavelets $\{\widetilde{\mathrm{up}}(\omega)\}$ is shown in Fig. 1c. Therefore, the condition of Theorem 2 is satisfied by construction. As proof, it is sufficient to verify equality (2), which holds if

$$|\widetilde{\mathrm{up}}(\omega)|^2 + |\widetilde{\mathrm{up}}(\omega - 2\pi)|^2 = 1 \qquad (8)$$

on the interval $\omega \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]$.

Let us prove Theorem 2 for the Kravchenko wavelets $\{\widetilde{\mathrm{up}}(\omega)\}$. The left-hand side of (8) is decomposed into a sum of up(•):

$$|\widetilde{\mathrm{up}}(\omega)|^2 + |\widetilde{\mathrm{up}}(\omega - 2\pi)|^2 = \mathrm{up}\left(\frac{3}{2\pi}\omega\right)$$
$$+ \mathrm{up}\left(\frac{3}{2\pi}\omega + 1\right) + \mathrm{up}\left(\frac{3}{2\pi}\omega - 1\right)$$
$$+ \mathrm{up}\left(\frac{3}{2\pi}(\omega - 2\pi)\right) + \mathrm{up}\left(\frac{3}{2\pi}(\omega - 2\pi) + 1\right)$$
$$+ \mathrm{up}\left(\frac{3}{2\pi}(\omega - 2\pi) - 1\right). \qquad (9)$$

Expanding the arguments in up(•), we obtain

$$|\widetilde{\mathrm{up}}(\omega)|^2 + |\widetilde{\mathrm{up}}(\omega - 2\pi)|^2 = \mathrm{up}\left(\frac{3}{2\pi}\omega\right) + \mathrm{up}\left(\frac{3}{2\pi}\omega + 1\right)$$
$$+ \mathrm{up}\left(\frac{3}{2\pi}\omega - 1\right) + \mathrm{up}\left(\frac{3}{2\pi}\omega - 3\right)$$
$$+ \mathrm{up}\left(\frac{3}{2\pi}\omega - 2\right) + \mathrm{up}\left(\frac{3}{2\pi}\omega - 4\right). \qquad (10)$$

Formula (10) can be written as

$$|\widetilde{\mathrm{up}}(\omega)|^2 + |\widetilde{\mathrm{up}}(\omega - 2\pi)|^2 = \sum_{n=-1}^{4} \mathrm{up}\left(\frac{3}{2\pi}\omega - n\right). \qquad (11)$$

According to property (5) of up(•),

$$|\widetilde{\mathrm{up}}(\omega)|^2 + |\widetilde{\mathrm{up}}(\omega - 2\pi)|^2$$
$$= \sum_{n=-1}^{4} \mathrm{up}\left(\frac{3}{2\pi}\omega - n\right)\Bigg|_{\omega \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]} = 1. \qquad (12)$$

Then

$$\sum_{n \in \mathbb{Z}} \left|\hat{\varphi}(\omega + 2\pi n)^2\right|_{\omega \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]}$$
$$= \sum_{n=-1}^{4} \mathrm{up}\left(\frac{3}{2\pi}\omega - n\right)_{\omega \in \left[\frac{2\pi}{3}, \frac{4\pi}{3}\right]} = 1, \qquad (13)$$

which was to be proved.

The sum of functions (13) is plotted in Fig. 2a.

4. For the third MRA property to be satisfied, the scaling equation

$$\hat{\varphi}(\omega) = H_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) \Leftrightarrow \hat{\varphi}(\omega) = \prod_{k=1}^{\infty} H_0\left(\frac{\omega}{2^k}\right) \qquad (14)$$

must hold.

The frequency response function of the scaling function $H_0(\omega)$ is determined from (14). Since $\hat{\varphi}\left(\frac{\omega}{2}\right) = 1$ on the interval $\left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$, and $\hat{\varphi}(\omega)$ vanishes outside $\left[-\frac{4\pi}{3}, \frac{4\pi}{3}\right]$, we have $H_0(\omega) = \hat{\varphi}(2\omega)$ if $\omega \in \left[-\frac{2\pi}{3}, \frac{2\pi}{3}\right]$ and $H_0(\omega) = 0$ if $\omega \in \left[-\pi, -\frac{2\pi}{3}\right] \cup \left[\frac{2\pi}{3}, \pi\right]$.

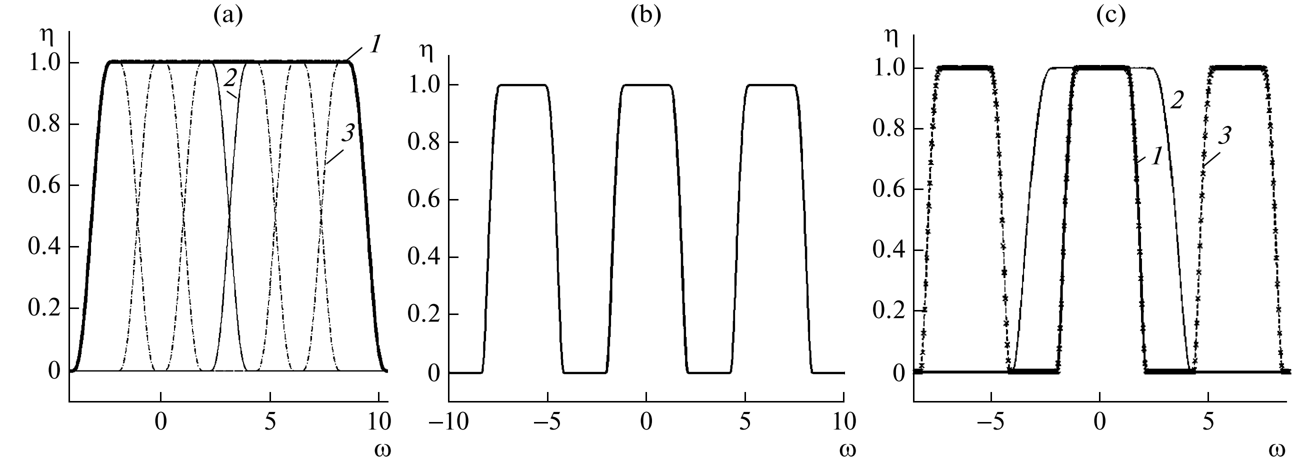

Fig. 2. Plots of functions: (a) $|\widetilde{\mathrm{up}}(\omega)|^2 + |\widetilde{\mathrm{up}}(\omega - 2\pi)|^2$: (*1*) $|\widetilde{\mathrm{up}}(\omega)|^2 + |\widetilde{\mathrm{up}}(\omega - 2\pi)|^2$; (*2*) $|\widetilde{\mathrm{up}}(\omega)|^2$ и $|\widetilde{\mathrm{up}}(\omega - 2\pi)|^2$; and (*3*) $\sum_{n=1}^{a} \mathrm{up}\left(\frac{3}{2\pi}\omega - n\right)$, $\alpha = -1, 0, \ldots, 4$; (b) $H_0(\omega)$ for the Kravchenko wavelet $\{\widetilde{\mathrm{up}}(\omega)\}$; and (c) verification of the scaling equation: (*1*) $H_0(\omega/2)\hat{\varphi}(\omega/2)$, (*2*) $\hat{\varphi}(\omega/2)$, and (*3*) $H_0(\omega/2)$.

Next, $H_0(\omega)$ is extended periodically with a period of 2π (Fig. 2b):

$$H_0(\omega) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(2(\omega + 2\pi n)). \quad (15)$$

Substituting (15) into (14) shows that the scaling equation holds (Fig. 2c):

$$H_0\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right) = \sum_{n \in \mathbb{Z}} \hat{\varphi}(\omega + 2\pi n)\hat{\varphi}\left(\frac{\omega}{2}\right) = \hat{\varphi}(\omega). \quad (16)$$

Then the following theorem [6, 7] is true for $H_0(\omega)$.

Theorem 3. *If the shifts $\varphi_n(x) = \varphi(x - n)$ of the scaling function $\varphi(x)$ form an orthonormal basis in V_0, then the frequency function $H_0(\omega)$ has the property*

$$|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 1. \quad (17)$$

The proof of Theorem 3 follows from (13) and (15). A geometric construction of (17) is presented in Fig. 3a. According to [6], the first two MRA properties, which are stated below as theorems, are also satisfied.

Theorem 4. *If the shifts $\varphi_n(x) = \varphi(x - n)$ of the scaling function $\varphi(x)$ form a Riesz basis in V_0, then $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$.*

Theorem 5. *Suppose that the scaling function $\varphi(x) \in L^2(\mathbb{R})$ satisfies condition* (1), *and let $\hat{\varphi}(\omega)$ be bounded for all ω and be continuous in the neighborhood of $\omega = 0$. Then $\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R})$.*

Thus, we have constructed the Fourier transform $\hat{\varphi}(\omega)$ of $\varphi(x)$ (Fig. 3b) generating MRA:

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega) e^{i\omega x} d\omega = \frac{1}{\pi}\int_{0}^{-4\pi/3} \hat{\varphi}(\omega)\cos\omega x\, d\omega. \quad (18)$$

Therefore, we can define an orthogonal basis obtained by contractions and shifts of the wavelet $\psi(x) \in W_0$ ($V_{j+1} = V_j \oplus W_j$, $W_j \perp V_j$, $W_j \perp W_k$ for all $j, k \in \mathbb{Z}$, $k \neq j$), which has many of the properties of $\varphi(x)$. According to [6, 7], the Fourier transform of $\hat{\psi}(\omega)$ (the Fig. 3c) is defined as

$$\hat{\psi}(\omega) = e^{i\omega/2}\overline{H_0\left(\frac{\omega}{2} + \pi\right)}\hat{\varphi}\left(\frac{\omega}{2}\right)$$

$$= e^{i\omega/2}(\hat{\varphi}(\omega - 2\pi) + \hat{\varphi}(\omega + 2\pi))\hat{\varphi}\left(\frac{\omega}{2}\right). \quad (19)$$

5. Let us show that the shifts $\psi_n(x) = \psi(x - n)$ form an orthonormal basis [8]:

$$\sum_{n \in \mathbb{Z}} |\hat{\psi}(\omega + 2\pi n)|^2 = \sum_{n \in \mathbb{Z}} |\hat{\psi}(\omega + 4\pi n)^2|$$

$$+ \sum_{n \in \mathbb{Z}} |\hat{\psi}(\omega + 2\pi + 4\pi n)|^2$$

$$= \left|H_0\left(\frac{\omega}{2} + \pi\right)\right|^2 \sum_{n \in \mathbb{Z}} \left|\hat{\varphi}\left(\frac{\omega}{2} + 2\pi n\right)\right|^2$$

$$+ \left|H_0\left(\frac{\omega}{2}\right)\right|^2 \sum_{n \in \mathbb{Z}} \left|\hat{\varphi}\left(\frac{\omega}{2} + \pi + 2\pi n\right)\right|^2$$

$$= \left|H_0\left(\frac{\omega}{2} + \pi\right)\right|^2 + \left|H_0\left(\frac{\omega}{2}\right)\right|^2 = 1. \quad (20)$$

Relations (14), (17), and (19) imply that the squares of the Fourier transform of wavelets and the scaling function

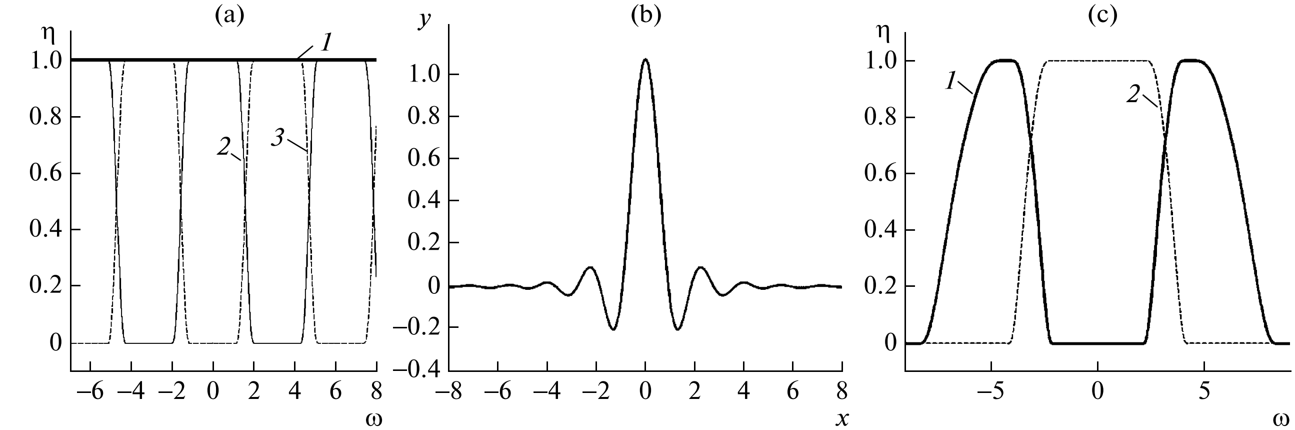

Fig. 3. (a) Verification of the condition $|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 1$: (*1*) $|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2$; (*2*) $|H_0(\omega)|^2$; and (*3*) $|H_0(\omega + \pi)|^2$; (b) plot of the scaling function $\varphi(x)$ of the Kravchenko wavelets $\{\widetilde{\text{up}}(\omega)\}$; and (c) $\hat{\psi}(\omega)$ for the Kravchenko wavelet $\{\widetilde{\text{up}}(\omega)\}$: (*1*) $\hat{\psi}(\omega)$ and (2) $\hat{\varphi}(\omega)$.

$$|\hat{\psi}(2\omega)|^2 = |H_0(\omega+\pi)|^2|\hat{\varphi}(\omega)|^2$$
$$= (1 + |H_0(\omega)|^2)|\hat{\varphi}(\omega)|^2 = |\hat{\varphi}(\omega)|^2 - |\hat{\varphi}(2\omega)|^2$$

have the property

$$|\hat{\varphi}(\omega)|^2 + |\hat{\psi}(\omega)|^2 = \left|\hat{\varphi}\left(\frac{\omega}{2}\right)\right|^2. \tag{21}$$

Substituting the Fourier transforms of the scaling function $\hat{\varphi}(\omega)$ and the mother wavelet $\hat{\psi}(\omega)$ into (21), we see that this property holds (Fig. 4a). This suggests that, using the Kravchenko wavelets $\{\widetilde{\text{up}}(\omega)\}$ and the scaling function of level j, we can obtain the scaling function of level $j + 1$ (the principle of embedded subspaces holds). Taking the inverse Fourier transform of $\hat{\psi}(\omega)$ yields a wavelet in the spatial domain of $\psi(x)$ (Fig. 4b):

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega/2}\hat{\varphi}\left(\frac{\omega}{2}\right)(\hat{\varphi}(\omega - 2\pi) + \hat{\varphi}(\omega + 2\pi))e^{i\omega x}d\omega$$
$$= \frac{1}{\pi}\int_{2\pi/3}^{-8\pi/3} \hat{\varphi}(\omega/2)\hat{\varphi}(\omega - 2\pi)\cos\omega(x + 0.5)d\omega. \tag{22}$$

The Kravchenko function $\psi(x)$ has all the properties required for a wavelet basis, specifically:

(i) $\psi_{jn}(x) \in L^2(\mathbb{R})$, $\|\psi_{jn}(x)\| = 1$.

(ii) The mean is zero: $\int_{-\infty}^{\infty}\psi_{jn}(x)dx = 0$ or, equivalently, $\hat{\psi}(0) = 0$.

(iii) For any scale j, the shifts $\psi_{jn}(x) = \psi_j(x - n)$ form an orthonormal basis.

The functions $\varphi(x)$ and $\psi(x)$ satisfy the scaling equations

$$\varphi(x) = \sqrt{2}\sum_{n \in \mathbb{Z}} h_n\varphi(2x - n),$$
$$\psi(x) = \sqrt{2}\sum_{n \in \mathbb{Z}} g_n\varphi(2x - n), \quad g_n = (-1)^{n+1}\bar{h}_{-n-1}, \tag{23}$$

where $\{h_n\}$ are the elements of the filter $H(\omega) = \sqrt{2}\,H_0(\omega) = \sum_{n \in \mathbb{Z}} h_n e^{-in\omega}$ (low-pass decomposition filter), which are found by expanding $H_0(\omega)$ in a Fourier series:

$$h_n = \frac{\sqrt{2}}{2\pi}\int_{-\pi}^{\pi} H_0(\omega)e^{in\omega}d\omega. \tag{24}$$

The coefficients of the filter $\{h_n\}$ (Table 1) satisfy the condition

$$\frac{1}{\sqrt{2}}\sum_{n \in \mathbb{Z}} h_n = H_0(0) = 1. \tag{25}$$

The other filters are defined in terms of $H(\omega)$ as follows. The high-pass decomposition filter is

$$G(\omega) = e^{i\omega}\overline{H(\omega + \pi)} = \sum_{n \in \mathbb{Z}} g_n e^{-in\omega}$$
$$= \sum_{n \in \mathbb{Z}} (-1)^{n+1}\bar{h}_{-n-1}e^{-i\omega n};$$

the low-pass reconstruction filter is

$$\tilde{H}(\omega) = \overline{H(\omega)} = \sum_{n \in \mathbb{Z}} \tilde{h}_n e^{-in\omega} = \sum_{n \in \mathbb{Z}} \bar{h}_{-n}e^{-in\omega};$$

and the high-pass reconstruction filter is

$$\tilde{G}(\omega) = e^{-i\omega}H(\omega+\pi) = \sum_{n\in\mathbb{Z}}\tilde{g}_n e^{-in\omega}$$

$$= \sum_{n\in\mathbb{Z}}(-1)^{n-1}h_{n-1}e^{-in\omega}.$$

The Kravchenko functions $\{\widetilde{\mathrm{up}}\,(\omega)\}$ $\varphi(x)$, and $\psi(x)$ are not compactly supported. However, they decrease rapidly (since they are infinitely differentiable). Therefore, effective supports $\mathrm{supp}(\varphi_{\mathrm{eff}}(x))$ and $\mathrm{supp}(\psi_{\mathrm{eff}}(x))$ can be determined by the conditions

$$\|\varphi-\varphi_{\mathrm{eff}}\|\times 100 \le 0.0001\%,$$
$$\|\psi-\psi_{\mathrm{eff}}\|\times 100 \le 0.001\%. \quad (26)$$

6. The supports of the new Kravchenko functions $\{\widetilde{\mathrm{up}}\,(\omega)\}$ have the form $\mathrm{supp}(\varphi_{\mathrm{eff}}(x))$ = [–8, 8] and $\mathrm{supp}(\psi_{\mathrm{eff}}(x))$ = [–8, 8]. Then the number of coefficients of the filter $\{h_n\}$ is bounded ($n = -16, \ldots, 16$). The Kravchenko wavelets $\{\widetilde{\mathrm{up}}\,(\omega)\}$ are similar in properties to the Meyer and Kotelnikov–Shannon wavelets, since their construction algorithms are similar. In this context, the former have the following advantages: they are symmetric and decrease much faster than the Meyer wavelets, but they have a noncompact support. In contrast to the Meyer wavelets, the functions $\hat{\varphi}\,(\omega)$ and $\hat{\psi}\,(\omega)$ for the Kravchenko wavelets $\{\widetilde{\mathrm{up}}\,(\omega)\}$ are written as a unified analytical expression and belong to the class of infinitely differentiable functions. The localization of the Kravchenko wavelets $\{\widetilde{\mathrm{up}}\,(\omega)\}$ in time and frequency is much better than that of the Kotelnikov–Shannon wavelets. A further comparison with available wavelets has to be performed in terms of effective support (26) and the uncertainty constants $\Delta_\varphi\Delta_{\hat{\varphi}}$ and $\Delta_\psi\Delta_{\hat{\psi}}$. For a function $f(t)$, the latter are defined according to [8] as

$$t_f^* = \frac{1}{\|f(t)\|^2}\int_{-\infty}^{\infty} t|f(t)|^2dt,$$
$$\Delta_f = \frac{1}{\|f(t)\|}\left(\int_{-\infty}^{\infty}(t-t_f^*)^2|f(t)|^2dt\right)^{1/2}, \quad (27)$$

Table 1. Coefficients of the filter h_n (24), (25) for the new scaling function $\varphi(x)$ of the Kravchenko wavelets $\{\widetilde{\mathrm{up}}(\omega)\}$

n	$h_n = h_{-n}$	n	$h_n = h_{-n}$
0	0.757698251288	17	0.000384982816
1	0.438708321041	18	0.000499860951
2	–0.047099287129	19	–0.000700388155
3	–0.118027008279	20	0.000468702885
4	0.037706980974	21	0.000255769244
5	0.043603935723	22	–0.000649033581
6	–0.025214528289	23	0.000266223602
7	–0.011459893503	24	0.000307507863
8	0.013002207742	25	–0.000463771747
9	–0.001878954975	26	0.000104807634
10	–0.003758906625	27	0.000324973138
11	0.005085949920	28	–0.000288500372
12	–0.001349824585	29	–0.000066833177
13	–0.003639380570	30	0.000021430184
14	0.002763059895	31	–0.000018524173
15	0.001188712844	32	–0.000032851429
16	–0.001940226446		

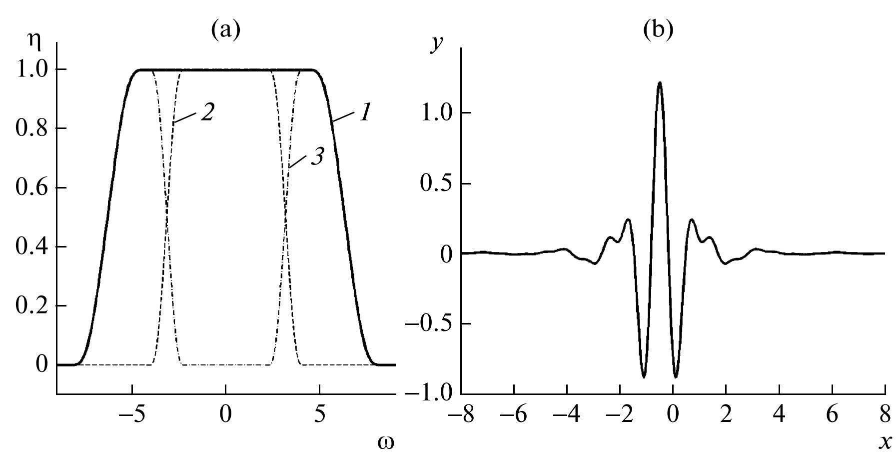

Fig. 4. (a) Verification of the condition $|\hat{\varphi}\,(\omega)|^2 + |\hat{\psi}\,(\omega)|^2 = |\hat{\varphi}\,(\omega/2)|^2$: (*1*) $|\hat{\varphi}\,(\omega)|^2 + |\hat{\psi}\,(\omega)|^2$; (*2*) $|\hat{\varphi}\,(\omega)|^2$; and (*3*) $|\hat{\psi}\,(\omega)|^2$. (b) Plot of $\psi(x)$ for the Kravchenko wavelets $\{\widetilde{\mathrm{up}}\,(\omega)\}$.

Table 2. Physical characteristics of various wavelet systems

Wavelets	$(\hat{\varphi})(\omega)$	$\mathrm{supp}(\varphi_{\mathrm{eff}}(x))$, $\mathrm{supp}(\psi_{\mathrm{eff}}(x))$	Uncertainty constants	
			$\Delta_\varphi \Delta_{\hat{\varphi}}$	$\Delta_\psi \Delta_{\hat{\psi}}$
Kravchenko $\{\widetilde{\mathrm{up}}(\omega)\}$	$\sqrt{\sum_{n=-1}^{1} \mathrm{up}\left(\frac{3}{2\pi}t + n\right)}$	[–8, 8]	0.87264	2.83008
Meyer	$\begin{cases} 1, -\frac{2\pi}{3} \le \lvert\omega\rvert \le \frac{2\pi}{3}, \\ \cos\left(\frac{\pi}{2}\nu\left(\frac{3}{2\pi}\lvert\omega\rvert - 1\right)\right), \frac{2\pi}{3} \le \lvert\omega\rvert \le \frac{4\pi}{3} \\ 0 \text{ otherwise } \omega \end{cases}$ $\nu(x) = x^4(35 - 84x + 70x^2 - 20x^3)$, $x \in [0, 1], \nu(x) + \nu(1 - x) = 1$	[–8, 8]	1.01148	3.27802
Kotelnikov–Shannon	$\begin{cases} 1, -\pi \le \lvert\omega\rvert \le \pi \\ 0 \text{ otherwise } \omega \end{cases}$	$[-10^4, 10^4]$	∞	∞

$$\omega_f^* = \frac{1}{\|\hat{f}(\omega)\|^2} \int_{-\infty}^{\infty} \omega |\hat{f}(\omega)|^2 d\omega,$$
$$\Delta_{\hat{f}} = \frac{1}{\|\hat{f}(\omega)\|} \left(\int_{-\infty}^{\infty} (\omega - \omega_f^*)^2 |\hat{f}(\omega)|^2 d\omega \right)^{1/2}. \quad (28)$$

The uncertainty constants are a major physical characteristic of wavelets, since they give complete information on the space-frequency resolution of a function and the distribution of its energy. An important point is that an uncertainty constant is invariant under shifts, contractions, and scaling of the function. As a result, this estimate is effective in the analysis of particular MRA schemes. The space-frequency localization is characterized by the products $\Delta_\varphi \Delta_{\hat{\varphi}}$ and $\Delta_\psi \Delta_{\hat{\psi}}$. Table 2 presents comparative physical characteristics for the Kravchenko $\{\widetilde{\mathrm{up}}(\omega)\}$, Meyer, and Kotelnikov–Shannon wavelets.

7. Thus, we have proposed and justified a new system of Kravchenko wavelets $\{\widetilde{\mathrm{up}}(\omega)\}$ that has all the properties of MRA. The Kravchenko wavelets were compared with other known wavelets in terms of physical characteristics. The new wavelets were shown to have a better time-frequency localization. The application of the Kravchenko wavelets in digital signal processing showed that they are effective in noise suppression and signal and image compression [6]. The theory of new wavelets will be developed as applied to continuous analysis, wavelet batch analysis, multidimensional wavelet transforms, digital signal and image processing, wavelet analysis of cardiac signals, radiolocation, acoustic optics, and boundary value problems in mathematical physics [6].

ACKNOWLEDGMENTS

This work was supported by the program "Leading Scientific Schools" within the project "New Methods Based on Atomic Functions, Wavelets, and Fractals for Some Problems in Acoustic Optics, Radio Physics, and Medicine," project no. NSh-5708.2008.9.

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