Choi–Williams Transform and Atomic Functions in Digital Signal Processing

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INTRODUCTION

In recent years, signals of new types are finding ever-widening applications in various areas of science and engineering [1]. For their analysis, it is expedient to apply, along with the conventional Fourier transform, new mathematical methods based on the wavelet transform, the theory of atomic functions (AF), and the Wigner transform [2–5]. On the other hand, in actual practice, signals are always detected and processed against the background of noise. External noise is most often represented by a random sequence of impulses with random parameters; moreover, it is not necessarily Gaussian. For such noise, expansion components of different scales turn out to be correlated and, because of this, linear methods of analysis are inefficient. The qualitative indices for the nonlinear filtering of noise can be increase by means of applying nonlinear transforms. Therefore, it is reasonable to study the appropriateness of applying the Choi–Williams transform in signal analysis and to compare it to the Wigner transform (WFT), and to AF.

BASIC PROPERTIES OF THE Choi–Williams TRANSFORM

The Choi–Williams transform [6, 7] of a signal \( f(t) \) has the form

\[
P_{\text{CW}}[f(t)] = P_{\text{CW}}(f, \omega) = \left[ \frac{\sigma}{4\pi} \right] \exp\left( -i\omega t \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left( -\frac{(u-t)^2}{4t^2} \right) f(u + \frac{t}{2}) f^*(u - \frac{t}{2}) du \, dt,
\]

where the asterisk denotes the operation of complex conjugation and \( \sigma > 0 \) is the scale coefficient. As \( \sigma \to \infty \), the Choi–Williams transform reduces to the Wigner transform [6, 7]. Let us mention the basic properties of the Choi–Williams transform [6]. The function of spectral density (FSD) \( P_{\text{CW}}(f, \omega) \) is real-valued; the arguments \( \tau \) and \( \omega \) are symmetric; the energy of the signal is concentrated in the neighborhood of the point \((t_0, \omega_0)\) whose length is equal to the frequency and time length of the signal itself; FSD \( P_{\text{CW}}(f, \omega) \) is invariant under the phase shift of signal \( f(t) \) by the value \( \varphi_0 \); for a time shift of signal \( f(t) \) by \( t_0 \), \( P_{\text{CW}}(f, \omega) \) is shifted along the time axis; and, for a frequency shift of \( \tilde{f}(\omega) \) by \( \omega_0 \), \( P_{\text{CW}}(f, \omega) \) is shifted along the frequency axis. The time scaling of the argument of signal \( f(t) \) by \( a \) leads to the scaling of the arguments of FSD \( P_{\text{CW}}(f, \omega) \) by \( a \) and \( 1/a \). The frequency scaling of the argument of the signal FSD \( \tilde{f}(\omega) \) by \( \beta \) leads to the scaling of the arguments of FSD \( P_{\text{CW}}(f, \omega) \) by \( 1/\beta \) and \( \beta \), time reversal, complex conjugation, energy conservation, and the presence of marginal distributions and the interference term

\[
I[f_1, f_2] = P_{\text{CW}}[f_1, f_2] + P_{\text{CW}}[f_2, f_1].
\]

It should be noted that, unlike the spectrogram and scalogram, it is merely a convention that \( P_{\text{CW}}(f, \omega) \) may be called the function of spectral density, because it is not positive and, hence, cannot represent the joint time-frequency density of a signal [6]. The use of the Choi–Williams transform in signal analysis is important for the following reasons. First, it provides fuller...
analysis of function $u_p(t)$: (a) signal; (b) FSD of the Wigner transform; (c), (d), (e) FSD of the Choi–Williams transform for $\sigma = (c) 1000$, (d) 10, (e) 0.01; and (f) Fourier spectrogram.

Fig. 1. Analysis of function $u_p(t)$: (a) signal; (b) FSD of the Wigner transform; (c), (d), (e) FSD of the Choi–Williams transform for $\sigma = (c) 1000$, (d) 10, (e) 0.01; and (f) Fourier spectrogram.
Another disadvantage is that, with a decrease in the level of interferential terms (decrease of $\sigma$), the local singularities of the processed signal are also suppressed; this reduces the time-frequency resolution. Therefore, it is advisable to combine the Wigner transform with the Choi–Williams transform.

It is also interesting to compare the potentialities of the Choi–Williams transform to those of the WFT, as well as to the WFT spectrogram. The WFT is a linear transform of a signal, which, unlike the Choi–Williams transform, is not invariant under phase, frequency, and time shifts. In addition, the time-frequency resolution of the WFT is determined by the characteristics of the spectral window and turns out to be lower than that of the Choi–Williams transform. The toll of this is the presence of interference in the FSD of the Choi–Williams transform. Similarly to the Choi–Williams transform, the spectrogram of the WFT is invariant under phase, frequency, and time shifts; however, unlike the former, it has no marginal distributions and has a lower time-frequency resolution. It is important that the spectrogram cannot be obtained from the Choi–Williams transform by means of passing to the limit as $\sigma \to 0$. Nevertheless, as $\sigma \to 0$, the shape of the FSD of the Choi–Williams transform resembles the spectrogram.

Thus, when applying a combination of the Wigner transform, the WFT spectrogram, and the Choi–Williams transform, the first transform makes it possible to establish the time-frequency localization of the processed signal with a rather high accuracy; the second transform allows eliminating false “signals” caused by the interference in the FSD of the Wigner transform; and the third transform enables one to observe the process of suppression of interference over time and to distinguish interferential terms from the local singularities of the signal, which can be lost in the spectrogram because of its insufficient time-frequency resolution. The Choi–Williams transform, which has a high time-frequency resolution, is expedient to be used in the analysis of new classes of signals [2, 3] based on AF, Kravchenko–Rvachev windows, R-functions, and wavelets, because such signals are well localized in the time domain.

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**REFERENCES**


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