ISSN 1028-3358, Doklady Physics, 2007, Vol. 52, No. 4, pp. 207–210. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © V.F. Kravchenko, O.V. Lazorenko, V.I. Pustovoĭt, L.F. Chernogor, 2007, published in Doklady Akademii Nauk, 2007, Vol. 413, No. 6, pp. 750–753.

**PHYSICS**

# Choi–Williams Transform and Atomic Functions in Digital Signal Processing

**V. F. Kravchenko[a], O. V. Lazorenko[b], Academician V. I. Pustovoĭt[c], and L. F. Chernogor[b]**

Received November 21, 2006

PACS numbers: 07.50.Qx, 43.60.Wy

**DOI:** 10.1134/S102833580704009X

## INTRODUCTION

In recent years, signals of new types are finding ever-widening applications in various areas of science and engineering [1]. For their analysis, it is expedient to apply, along with the conventional Fourier transform, new mathematical methods based on the wavelet transform, the theory of atomic functions (AF), and the Wigner transform [2–5]. On the other hand, in actual practice, signals are always detected and processed against the background of noise. External noise is most often represented by a random sequence of impulses with random parameters; moreover, it is not necessarily Gaussian. For such noise, expansion components of different scales turn out to be correlated and, because of this, linear methods of analysis are inefficient. The qualitative indices for the nonlinear filtering of noise can be increase by means of applying nonlinear transforms. Therefore, it is reasonable to study the appropriateness of applying the Choi–Williams transform in signal analysis and to compare it to the Wigner transform, to the conventional windowed Fourier transform (WFT), and to AF.

## BASIC PROPERTIES OF THE CHOI–WILLIAMS TRANSFORM

The Choi–Williams transform [6, 7] of a signal $f(t)$ has the form

$$P_{CW}[f(t)] \equiv P_{CW}f(\tau, \omega) = \sqrt{\frac{\sigma}{4\pi}}\int_{-\infty}^{\infty}\frac{\exp(-i\omega t)}{|t|} \times \int_{-\infty}^{\infty}\exp\left(\frac{-(u-\tau)^2\sigma}{4t^2}\right)f\left(u+\frac{t}{2}\right)f^*\left(u-\frac{t}{2}\right)du\,dt, \quad (1)$$

where the asterisk denotes the operation of complex conjugation and $\sigma > 0$ is the scale coefficient. As $\sigma \to \infty$, the Choi–Williams transform reduces to the Wigner transform [6, 7]. Let us mention the basic properties of the Choi–Williams transform [6]. The function of spectral density (FSD) $P_{CW}f(\tau, \omega)$ is real-valued; the arguments $\tau$ and $\omega$ are symmetric; the energy of the signal is concentrated in the neighborhood of the point $(t_0, \omega_0)$ whose length is equal to the frequency and time length of the signal itself; FSD $P_{CW}f(\tau, \omega)$ is invariant under the phase shift of signal $f(t)$ by the value $\varphi_0$; for a time shift of signal $f(t)$ by $t_0$, $P_{CW}f(\tau, \omega)$ is shifted along the time axis; and, for a frequency shift of $\hat{f}(\omega)$ by $\omega_0$, $P_{CW}f(\tau, \omega)$ is shifted along the frequency axis. The time scaling of the argument of signal $f(t)$ by $a$ leads to the scaling of the arguments of FSD $P_{CW}f(\tau, \omega)$ by $a$ and $1/a$. The frequency scaling of the argument of the signal FSD $\hat{f}(\omega)$ by $\beta$ leads to the scaling of the arguments of FSD $P_{CW}f(\tau, \omega)$ by $1/\beta$ and $\beta$, time reversal, complex conjugation, energy conservation, and the presence of marginal distributions and the interference term

$$I[f_1, f_2] = P_{CW}[f_1, f_2] + P_{CW}[f_2, f_1].$$

It should be noted that, unlike the spectrogram and scalogram, it is merely a convention that $P_{CW}f(\tau, \omega)$ may be called the function of spectral density, because it is not positive and, hence, cannot represent the joint time-frequency density of a signal [6]. The use of the Choi–Williams transform in signal analysis is important for the following reasons. First, it provides fuller

[a] Institute of Radio Engineering and Electronics, Russian Academy of Sciences, Building 7, ul. Mokhovaya 11, GSP-9, Moscow K-9, 125009 Russia
[b] Karazin Kharkiv National University, pl. Svobody 4, Kharkiv, 61077 Ukraine
[c] Scientific and Technological Center of Unique Instrumentation, Russian Academy of Sciences, ul. Butlerova 15, Moscow, 117342 Russia
e-mail: kvf@pochta.ru; Leonid.F.Chernogor@univer.kharkov.ua

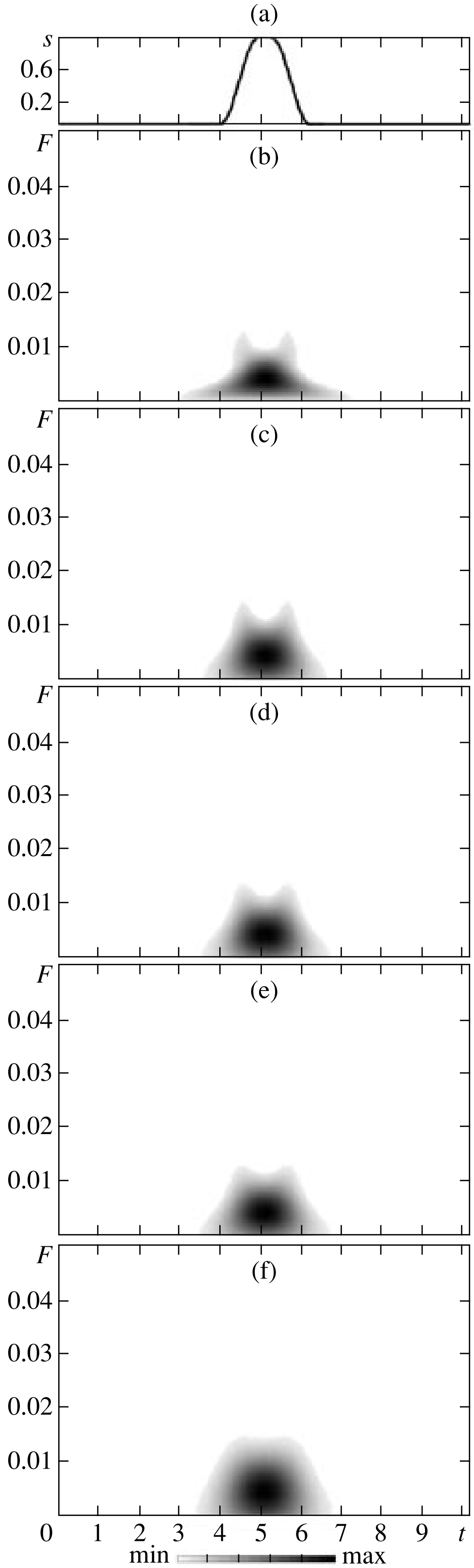


**Fig. 1.** Analysis of function up($t$): (a) signal; (b) FSD of the Wigner transform; (c), (d), (e) FSD of the Choi–Williams transform for $\sigma$ = (c) 1000, (d) 10, (e) 0.01; and (f) Fourier spectrogram.

information about the signal, rather than the employment of other tools such as spectrograms; second, it enables one to analyze the time-frequency content of individual singularities of the signal; and finally, it makes it possible to analyze the corresponding FSD of the signal.

## INVESTIGATION OF SIGNALS ON THE BASIS OF ATOMIC FUNCTIONS

Analytic expressions for the FSD $P_{CW}f(\tau, \omega)$ of the Choi–Williams transform can hardly be calculated; therefore, satisfactory results can be obtained only in the simplest cases. In practice, the Choi-Williams transform is most commonly performed by means of numerical calculations using computer algebra systems. In our calculations, we used the TimeFreq Toolbox for MATLAB 7.X [8]. We took AF and windows based on these AF [2, 3] as the analyzed signals. AF are well localized in both time and frequency domains. The Choi–Williams transform possesses good time-frequency properties. It is interesting to see whether this transform adequately reflects the local singularities of AF. Examples of analysis of AF are presented in Figs. 1 and 2. The FSD of the Choi–Williams transform of the parent AF up($x$) (of a typical video signal) has a typical shape with a dip in the middle. With decreasing $\sigma$, it becomes less pronounced. The spectrogram exhibits no such dip. For the Kravchenko–Rvachev window [2, 3] $w_2(x) = \mathrm{up}(x) + 0.01\mathrm{up}''(x)$, this typical dip can be seen only in the FSD of the Wigner transform $P_V f(\tau, \omega)$ (the primes denote the derivatives). For up'($x$) (a model ultrawideband signal), the FSD of the Choi–Williams transform (Fig. 2) has two pronounced localized formations (as typical as in the case of function up'($x$)) which are parts of a more complex structure. Two maximums located between these formations are of interferential nature. The FSD of the Choi–Williams transform of up''($x$) has an even more complex structure. In addition, a number of other AF were analyzed: $\mathrm{fup}_1(x)$, $\mathrm{fup}_2(x)$, $h_3(x)$, $h_4(x)$, $\Xi_2(x)$, and $\Xi_3(x)$.

## DISCUSSION AND CONCLUSIONS

The Choi–Williams transform has a number of advantages such as high time-frequency resolution; invariance under phase, frequency, and time shifts; and the presence of marginal distributions. Moreover, this transform (which is quadratic in the processed signal) provides a method of nonlinear signal processing. It is efficient, for instance, in the case where noise is not normally distributed. Parameter $\sigma$, which allows adjusting the level of interferential terms, makes it possible to avoid the interference between the signals or between the signal and noise; this interference is typical of the Wigner transform [5]. A drawback of the Choi–Williams transform is the complexity of integral (1), which

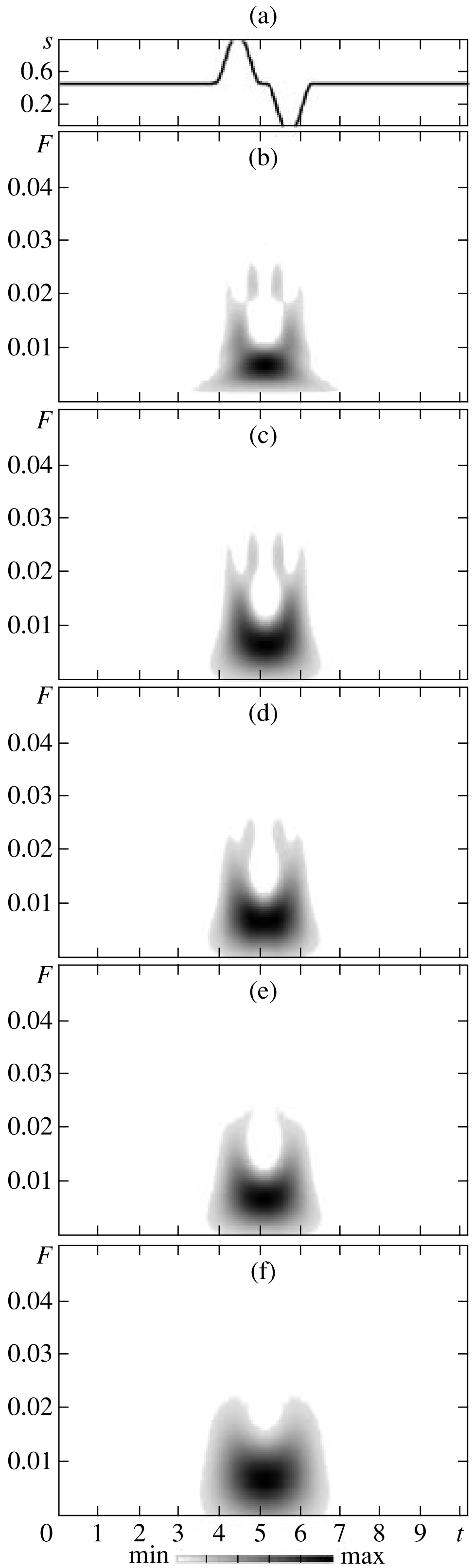


**Fig. 2.** Analysis of function up'($t$): (a) signal; (b) FSD of the Wigner transform; (c), (d), (e) FSD of the Choi–Williams transform for $\sigma$ = (c) 1000, (d) 10, (e) 0.01; and (f)—Fourier spectrogram.

makes it hardly applicable for analytic calculations. Another disadvantage is that, with a decrease in the level of interferential terms (decrease of $\sigma$), the local singularities of the processed signal are also suppressed; this reduces the time-frequency resolution. Therefore, it is advisable to combine the Wigner transform with the Choi–Williams transform.

It is also interesting to compare the potentialities of the Choi–Williams transform to those of the WFT, as well as to the WFT spectrogram. The WFT is a linear transform of a signal, which, unlike the Choi–Williams transform, is not invariant under phase, frequency, and time shifts. In addition, the time-frequency resolution of the WFT is determined by the characteristics of the spectral window and turns out to be lower than that of the Choi–Williams transform. The toll of this is the presence of interference in the FSD of the Choi–Williams transform. Similarly to the Choi–Williams transform, the spectrogram of the WFT is invariant under phase, frequency, and time shifts; however, unlike the former, it has no marginal distributions and has a lower time-frequency resolution. It is important that the spectrogram cannot be obtained from the Choi–Williams transform by means of passing to the limit as $\sigma \to 0$. Nevertheless, as $\sigma \to 0$, the shape of the FSD of the Choi–Williams transform resembles the spectrogram.

Thus, when applying a combination of the Wigner transform, the WFT spectrogram, and the Choi–Williams transform, the first transform makes it possible to establish the time-frequency localization of the processed signal with a rather high accuracy; the second transform allows eliminating false "signals" caused by the interference in the FSD of the Wigner transform; and the third transform enables one to observe the process of suppression of interference over time and to distinguish interferential terms from the local singularities of the signal, which can be lost in the spectrogram because of its insufficient time-frequency resolution. The Choi–Williams transform, which has a high time-frequency resolution, is expedient to be used in the analysis of new classes of signals [2, 3] based on AF, Kravchenko–Rvachev windows, R-functions, and wavelets, because such signals are well localized in the time domain.

The results of this work were partly presented at the Third International Conference "Ultrawideband and Ultrashort Impulse Signals," September 18–22, 2006, Sevastopol, Ukraine [9].

## REFERENCES

1. H. F. Harmuth, *Nonsinusoidal Waves for Radar and Radio Communication* (Academic, New York, 1981; Radio i Svyaz', Moscow, 1985).

2. V. F. Kravchenko and V. L. Rvachev, *Algebra of Logic, Atomic Functions and Wavelets in Physical Applications* (Fizmatlit, Moscow, 2006) [in Russian].

3. V. F. Kravchenko, *Lectures on the Theory of Atomic Functions and Some of Their Applications* (Radiotekhnika, Moscow, 2003) [in Russian].

4. O. V. Lazorenko, S. V. Lazorenko, and L. F. Chernogor, Élektromagn. Volny Élektron. Sist. **9** (9/10), 31 (2004).

5. O. V. Vishnivetskiĭ, V. F. Kravchenko, O. V. Lazorenko, and L. F. Chernogor, Élektromagn. Volny Élektron. Sist. **11** (6), 26 (2006).

6. L. Cohen, Proc. IEEE **77**, 941 (1989).

7. H.-J. Choi and W. J. Williams, IEEE Trans. Acoust., Speech, Signal Process. **37**, 862 (1989).

8. V. D'yakonov, *MATLAB 6: School-Book* (Piter, St. Petersburg, 2001) [in Russian].

9. L. F. Chernogor, V. F. Kravchenko, and O. V. Lazarenko, in *Proceedings of the III International Conference on Ultrawideband and Ultrashort Impulse Signals, Sevastopol, Ukraine, 2006*, pp. 32–37.

*Translated by A. Pankrat'ev*