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THEORETICAL PHYSICS

Analytical Two-Dimensional WA Systems of Kravchenko–Rvachev Functions and Their Physical Properties

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INTRODUCTION

In this paper, based on the theory of atomic functions and R-functions, we, for the first time, construct a novel class of analytical two-dimensional wavelet Kravchenko–Rvachev (2DWA KR) functions with reference regions of a nonstandard shape [1–5]. It is well-known [6] that these reference regions are intrinsic to problems associated with the space–time processing of experimental data, multidimensional digital filtering, and spectral analysis of signals having different physical natures. The present study consists of three parts. In the first part, we substantiate the method describing complicated-shape loci on the basis of the theory of R-functions (Rvachev functions) [1–3] of complicated form. In the second part, we propose and substantiate new relationships that allow us to construct analytical 2DWA KR functions. In the third part, physical results of a numerical experiment are analyzed with allowance for the results of [7–10].

ANALYTICAL DESCRIPTION OF COMPLICATED-SHAPE LOCI BY RVACHEV FUNCTIONS

Solving the inverse problem of analytical geometry. Let in space R_2, region Ω be given with the piecewise-smooth boundary $\partial\Omega$. It is required to construct the function $\omega(x_1, x_2)$ positive inside Ω, negative outside Ω, and equal to zero at $\partial\Omega$. The equation $\omega(x_1, x_2) = 0$ determines the locus for points belonging to the boundary of the region [1, 2]. We denote the characteristic function corresponding to the region Ω as $\chi = (\omega(x_1, x_2) \geq 0)$. Then, having a certain system $\chi_i = (\omega_i(x_1, x_2) \geq 0)$, $i = 1, 2, \ldots, N$ of characteristic functions and the Boolean function $F(X_1, X_2, \ldots, X_N)$, we can construct the predicate $\chi = F(\chi_1, \chi_2, \ldots, \chi_N) = F((\omega_1 \geq 0), \ldots, (\omega_N \geq 0))$. The region is constructed on the basis of auxiliary regions $\Omega_1, \Omega_2, \ldots, \Omega_n$, with rules of the Boolean algebra being used. Hence, the following definition holds true [1, 2].

Definition 1. The function whose sign is entirely determined by signs of its arguments, we call the R-function corresponding to the partition of the number axis within the interval $(-\infty, 0)$ and $[0, \infty)$.

Definition 2. The system of functions H, which is composed of R-functions, is called sufficiently complete provided that the set of all superpositions of elements of H (the set of H-realized functions) has a nonempty intersection with each branch of the set of R-functions.

Definition 3. In accordance with [2], the locus described by the equation $f(x) = 0$ is the set of all points in the space E^n for which the function $y = f(x)$, $x = x(x_1, x_2, \ldots, x_n)$ takes zero values.

Hence, the theory of R-functions makes it possible to describe at the analytical level loci of a complicated shape. The complicated geometric structures may be considered as composed of simpler ones. They form a system of reference geometric objects for which there are the defining equations $f = 0$ (or inequalities $f \geq 0$). Thus, the following theorem is valid [2].

Theorem 1. If f is a continuous function defined in the space E^n, then the locus $f = 0$ and the region $f \geq 0$ are closed sets (the empty set and the entire space are considered to be simultaneously open and closed).

Thus, there exists a certain finite system of reference objects $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$, with $F_1(X_1, X_2, \ldots, X_n)$ being an arbitrary Boolean function that can be represented in the form of a superposition of several operations: conjunction, disjunction, and negation. The object $\Sigma(x)$ obtained can be considered as a result of the application (to reference objects $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$) of a

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Table 1. Basic systems of R-functions

System	Accompanying Boolean function		
	$F_1(a, b) = a \wedge b$	$F_2(a, b) = a \vee b$	$F_3(b) = \overline{b}$
R_α	$\dfrac{x_1 + x_2 - \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}}{1+\alpha}$	$\dfrac{x_1 + x_2 + \sqrt{x_1^2 + x_2^2 - 2\alpha x_1 x_2}}{1+\alpha}$	$\overline{x} = -x$
R_0^m	$\left(x_1 + x_2 - \sqrt{x_1^2 + x_2^2}\right) \cdot (x_1^2 + x_2^2)^{m/2}$	$\left(x_1 + x_2 + \sqrt{x_1^2 + x_2^2}\right) \cdot (x_1^2 + x_2^2)^{m/2}$	$\overline{x} = -x$
R_1	$\frac{1}{2}(x + y - \lvert x - y \rvert)$	$\frac{1}{2}(x + y + \lvert x - y \rvert)$	$\overline{x} = -x$

certain set of operations $X_1 \cap X_2$, $X_1 \cup X_2$, and $\overline{X}$ (Table 1).

We now consider examples of the description of complicated-shape regions.

Example 1. The asterisk-shaped reference region (Fig. 1a). The logic formula for the description of this region is of the form

$$\omega_1(x_1, x_2) = [\omega_{11} \wedge \omega_{12} \wedge \omega_{13}] \vee [\omega_{14} \wedge \omega_{15} \wedge \omega_{16}], \quad (1)$$

where

$$\omega_{11} = \frac{-1}{\sqrt{3}}\left(x_1 - \sqrt{3}\right), \quad \omega_{12} = 1 + \frac{1}{3}\left(x_1 + x_2\sqrt{3}\right),$$

$$\omega_{13} = 1 + \frac{1}{3}\left(x_1 - x_2\sqrt{3}\right), \quad \omega_{14} = \frac{1}{\sqrt{3}}\left(x_1 + \sqrt{3}\right),$$

$$\omega_{15} = 1 + \frac{-1}{3}\left(x_1 - x_2\sqrt{3}\right), \quad \omega_{16} = 1 + \frac{-1}{3}\left(x_1 + x_2\sqrt{3}\right).$$

The behavior of the function $\omega_1(x_1, x_2)$ is shown in Fig. 1b.

Example 2. The complicated reference region is obtained as a combination of a circle and a cross (Fig. 1c). The logic formula for the description of the reference region can be written out in the form

$$\omega_2(x_1, x_2) = [\omega_{21} \wedge \omega_{22} \wedge \omega_{23} \wedge \omega_{24}] \vee [\omega_{25} \wedge \omega_{26} \wedge \omega_{27} \wedge \omega_{28}] \vee \omega_{29}, \quad (2)$$

where

$$\omega_{21} = -2x_1 + 1, \quad \omega_{22} = 2x_1 + 1, \quad \omega_{23} = 0.4x_2 + 1,$$

$$\omega_{24} = -0.4x_2 + 1, \quad \omega_{25} = -2x_2 + 1, \quad \omega_{26} = 2x_2 + 1,$$

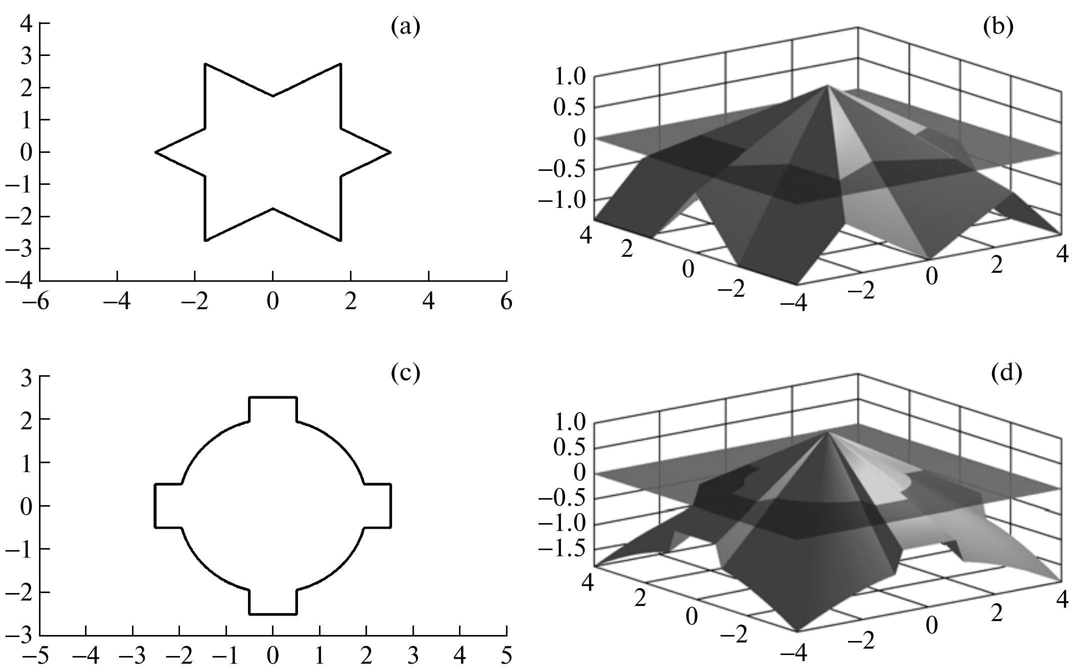

Fig. 1. (a, c) Reference areas of complex geometry and (b, d) the corresponding R-functions and planes $y = 0$.

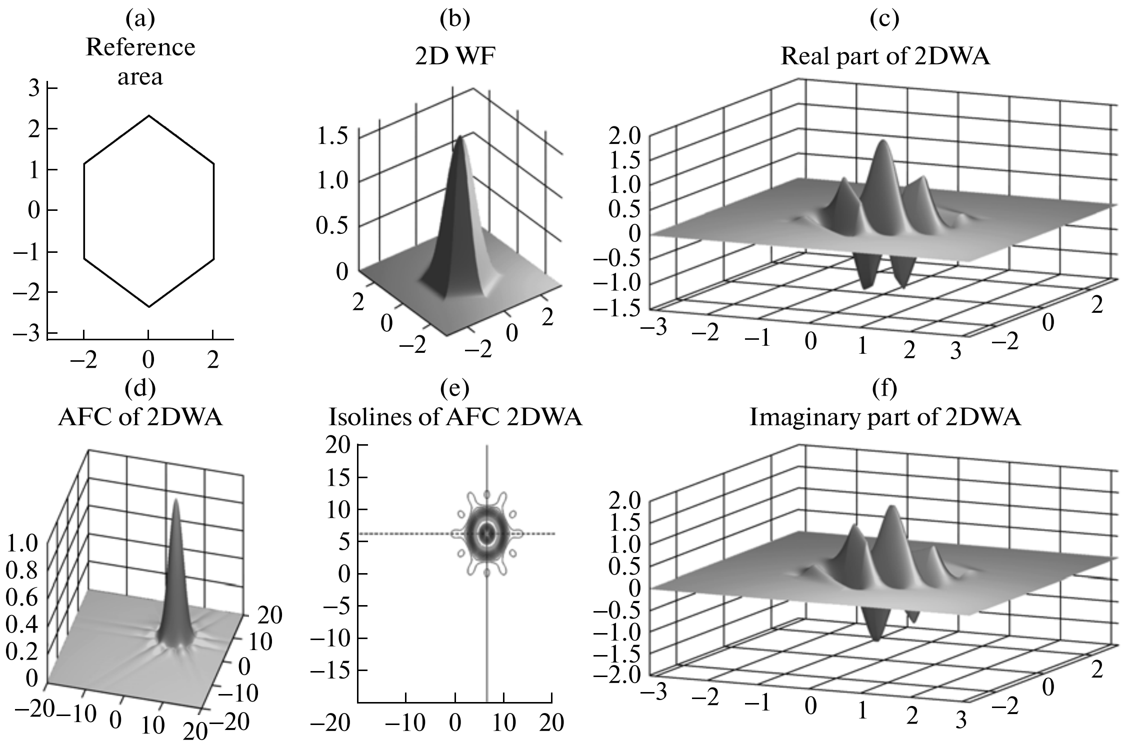

Fig. 2. (a) Reference area 1; (b) 2D wavelet functions $s(x, y)$; (c) real-valued and (f) imaginary parts of the constructed wavelet function; (d) wavelet spectrum and (e) its isolines; $\eta_1 = \eta_2 = 2\pi$. AFC stands for Amplitude-frequency characteristic.

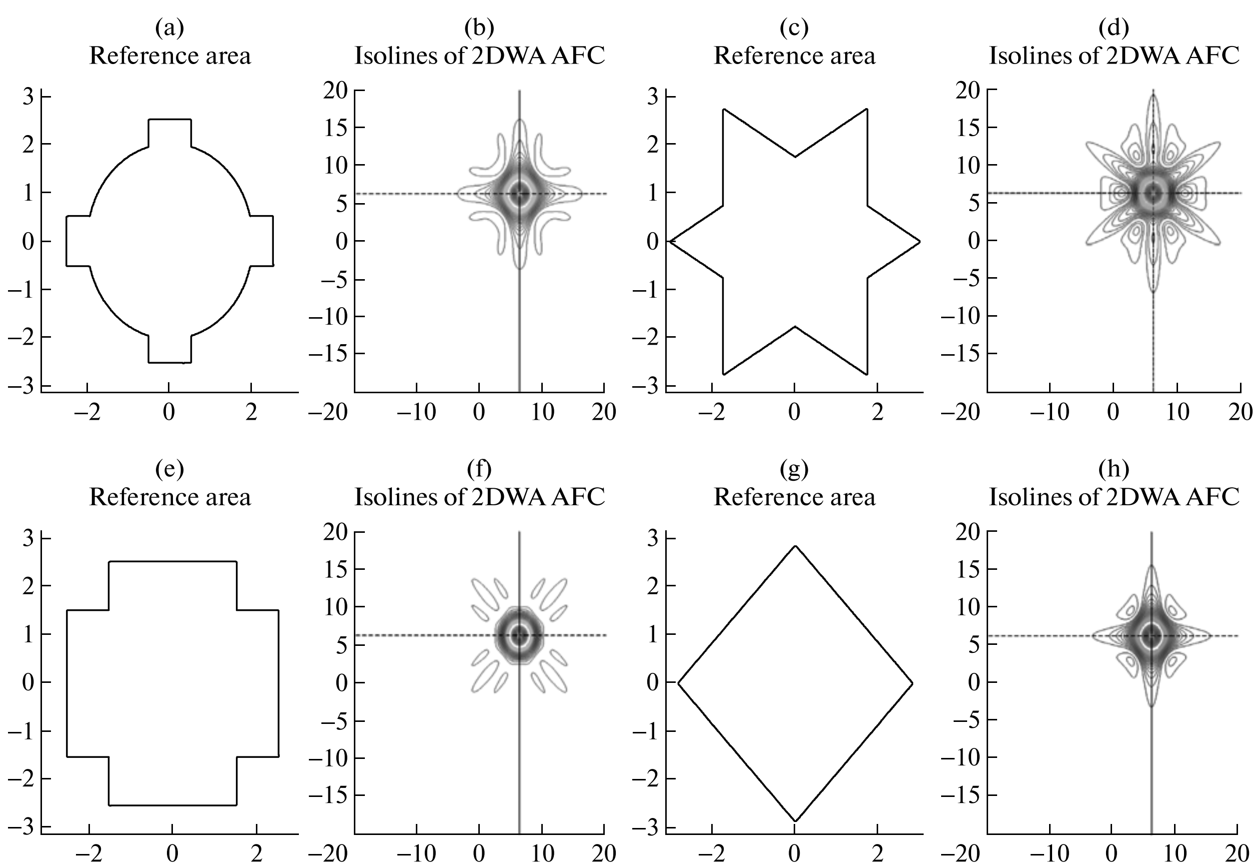

Fig. 3. (a, c, e, g) Reference areas 2–5; (b, d, f, h) isolines for spectra of the corresponding wavelets; $\eta_1 = \eta_2 = 2\pi$.

Table 2. Physical characteristics of constructed wavelet functions ($\eta_1 = \eta_2 = 2\pi$)

Number of a reference area	$\frac{\gamma_4}{\gamma_3}$	$\frac{\gamma_2}{\gamma_3}$	γ_7	γ_8	γ_9	Δ	J
Atomic function $\mathrm{fup}_3(x)$							
1	1.375	2.875	0.326	0.312	−33.436	0.204	1.855
2	1.429	2.286	0.308	0.295	−30.503	0.268	1.922
3	1.429	2.857	0.261	0.250	−21.543	0.310	2.309
4	1.500	2.667	0.215	0.206	−30.542	0.187	2.224
5	1.429	2.143	0.282	0.270	−27.838	0.202	1.919
Atomic function $\Xi_3(x)$							
1	1.429	3.000	0.634	0.271	−32.059	0.175	1.780
2	1.286	2.000	0.600	0.256	−29.046	0.236	1.660
3	1.500	3.000	0.507	0.217	−20.995	0.267	2.202
4	1.333	2.333	0.418	0.179	−29.911	0.161	1.915
5	1.500	2.167	0.549	0.235	−26.663	0.174	1.829
Atomic function $h_3(x)$							
1	1.200	2.000	0.215	0.121	−20.365	0.092	2.344
2	1.500	2.000	0.203	0.114	−17.362	0.150	2.728
3	1.500	2.750	0.172	0.097	−15.036	0.144	3.197
4	1.250	2.000	0.142	0.080	−21.662	0.087	3.011
5	1.500	2.000	0.186	0.104	−16.463	0.092	2.827

$$\omega_{27} = 0.4x_1 + 1, \quad \omega_{28} = -0.4x_1 + 1,$$

$$\omega_{29} = 1 - \sqrt{\frac{x_1^2}{4} + \frac{x_2^2}{4}}.$$

The shape of the function $\omega_2(x_1, x_2)$ is shown in Fig. 1d.

ANALYTICAL TWO-DIMENSIONAL KRAVCHENKO–RVACHEV WAVELETS

Construction of 2DWA KR functions consists of several stages. At the first stage, the locus of the reference region is given whether in the time space or in the frequency space. Then, the equation $\omega(x, y) \geq 0$ for the reference area is composed on the basis of the complete system of *R*-functions. This equation satisfies the condition $\max \omega(x, y) = \omega(0,0) = 1$. At the second stage, the function

$$s(x, y) \equiv bw(c[1 - \omega(x, y)]),$$

is determined, where $w(*)$ is the one-dimensional weight function, whereas the quantities b, c are used for the normalization of the atomic function: $\max bw(cx) = bw(0) = 1$, $bw(c) = 0.5$. The function $s(x, y)$ is modulated by the complex exponent

$$\varphi(x, y) = s(x, y)e^{i(x\eta_1 + y\eta_2)}.$$

The parameters η_1, η_2 are chosen with allowance for the required frequency properties of the wavelet to be determined. The wavelet spectrum is calculated in accordance with the relationship

$$\hat{\varphi}(f_1, f_2) = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \varphi(x, y) e^{-i(f_1 x + f_2 y)} dx dy. \quad (3)$$

Thus, we have obtained the analytical 2DWA KR functions of complicated geometry. Constructing 2DWA KR functions with the reference region *1* is shown in Fig. 2. Figure 3 demonstrates reference areas *2–5*, as well as isolines for spectra of the corresponding wavelets.

PHYSICAL PROPERTIES OF ANALYTICAL TWO-DIMENSIONAL KRAVCHENKO–RVACHEV WAVELETS

In order to study two-dimensional functions, we make use of the modified physical characteristics [1, 2].

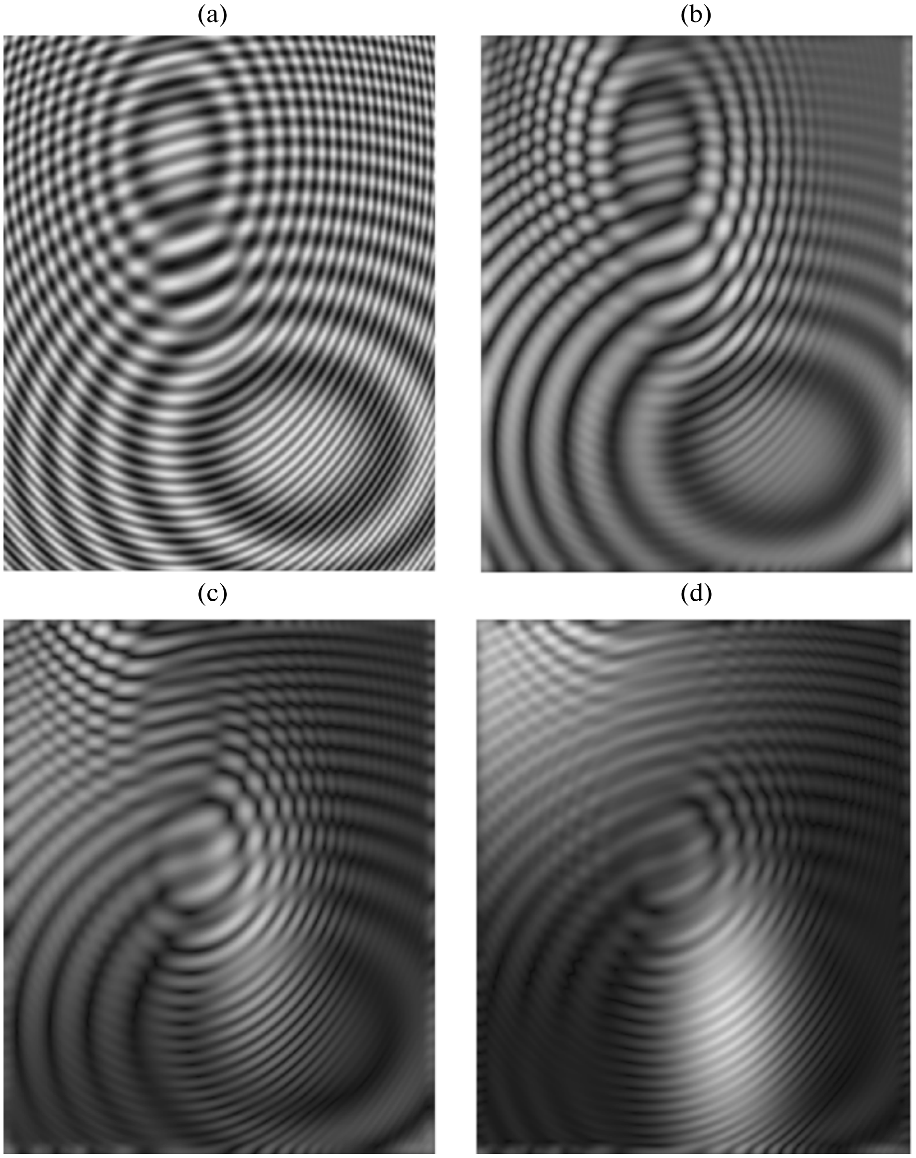

Fig. 4. (a) Sum of two spatial linearly frequency-modulated signals and results of the wavelet transformation for (b) $\eta_1 = \pi$, $\eta_2 = 0.5\pi$; (c) $\eta_1 = \pi$, $\eta_2 = \pi$, and (d) $\eta_1 = \pi$, $\eta_2 = 1.5\pi$.

The relative width of the spectral-density function (SDF) at the –6-dB level is $\left(\gamma_{4/3} = \frac{\gamma_4}{\gamma_3}\right)$, where γ_3 is the SDF width at the –3-dB level. The relative position of the first SDF zero is $\left(\gamma_{2/3} = \frac{\gamma_2}{\gamma_3}\right)$. The coherent amplification is described by the relationship

$$\gamma_7 = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x,y)|\,dxdy\right)^{-1}.$$

The equivalent noise band is given by the relation

$$\gamma_8 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x,y)|^2\,dxdy}{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x,y)|\,dxdy\right)^2} = \gamma_7^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\varphi(x,y)|^2\,dxdy.$$

The maximum level of side lobes (which is expressed in dB) is calculated according to the relationship

$$\gamma_9 = 10\log\max_k\left|\frac{\hat{\varphi}(f_{1m,k}, f_{2m,k})}{\hat{\varphi}(f_{1\max}, f_{2\max})}\right|^2,$$

where $(f_{1m,k}, f_{2m,k})$ are the coordinates of SDF local maxima, which differ from the coordinate $(f_{1\max}, f_{2\max})$ of the principal maximum. The constants of uncertainty for time and frequency are calculated by the formulas

$$x^* = \frac{1}{\|\varphi(x,y)\|^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x|\varphi(x,y)|^2 \, dxdy,$$

$$\Delta_x = \frac{1}{\|\varphi(x,y)\|}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-x^*)^2 |\varphi(x,y)|^2 \, dxdy\right)^{1/2},$$

$$f_1^* = \frac{1}{\|\hat{\varphi}(f_1,f_2)\|^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1|\hat{\varphi}(f_1,f_2)|^2 \, df_1 df_2,$$

$$\Delta_{f_1} = \frac{1}{\|\hat{\varphi}(f_1,f_2)\|}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(f_1-f_1^*\right)^2 |\hat{\varphi}(f_1,f_2)|^2 \, df_1 df_2\right)^{1/2}.$$

The values of Δ_y and Δ_{f_2} are calculated in a similar manner. The time–frequency localization is characterized by the products $\Delta = \Delta_x\Delta_{f_2} = \Delta_y\Delta_{f_1}$. The values of the calibration characteristics are

$$\gamma_{4/3}^* = 1, \quad \gamma_{2/3}^* = 1, \quad \gamma_7^* = 1, \quad \gamma_8^* = 1. \tag{4}$$

The closeness functional for a filter with respect to the perfect one. Using the results of [7–10], we construct the quality functional for the choice of analytical 2DWA KR functions and perform the analysis of the variation of physical characteristics for two-dimensional functions with respect to calibration functions (4). The functional of the filter closeness with respect to the perfect filter can be written in the form

$$J = \left|\frac{\gamma_{4/3}^* - \gamma_{4/3}}{\gamma_{4/3}^*}\right| + \left|\frac{\gamma_{2/3}^* - \gamma_{2/3}}{\gamma_{2/3}^*}\right| + \left|\frac{\gamma_7^* - \gamma_7}{10\gamma_7^*}\right| + \left|\frac{\gamma_8^* - \gamma_8}{10\gamma_8^*}\right| + \left|\frac{10}{\gamma_9}\right| + \Delta, \tag{5}$$

where $\gamma*$ is the calibration characteristic. In Table 2, physical characteristics and values for the functional of the filter closeness to the perfect filter for 2DWA KR functions synthesized with reference areas *1–5* and for the atomic function $\mathrm{fup}_3(x)$, $\Xi_3(x)$, $h_3(x)$ are given.

Numerical experiment. In order to illustrate the application of 2DWA KR functions, we consider the problem of the analysis of two spatial linearly frequency-modulated signals (Fig. 4a). The results of the two-dimensional wavelet transformation for different combinations $((\eta_1, \eta_2))$ are presented in Figs. 4b, 4c, and 4d.

Thus, we have constructed novel analytical two-dimensional wavelets on the basis of the theory of atomic functions and *R*-functions. As was verified by the numerical experiment, these functions can find wide application in solving problems of interference effects under the conditions of complicated geometry and in the case of signals and image recognition.

The results of this study were reported in part at the International Conference "Days on Diffraction Annual" [11].

REFERENCES

1. V. F. Kravchenko, *Lectures on Theory of Atomic Functions and Their Applications* (Radiotekhnika, Moscow, 2003) [in Russian].
2. V. F. Kravchenko and V. L. Rvachev, *Boolean Algebra, Atomic Functions and Wavelets in Physical Applications* (Fizmatlit, Moscow, 2006) [in Russian].
3. V. F. Kravchenko, O. S. Labun'ko, A. M. Lerer, and G. P. Sinyavskii, *Numerical Methods in Modern Radio Physics* (Fizmatlit, Moscow, 2009) [in Russian].
4. *Digital Signal and Image Processing: in Radio Physics Applications*, Ed. by V. F. Kravchenko (Fizmatlit, Moscow, 2007) [in Russian].
5. V. F. Kravchenko, H. M. Perez-Meana, and V. I. Ponomaryov, *Adaptive Digital Processing of Multidimensional Signals with Applications* (Fizmatlit, Moscow, 2009).
6. D. E. Dudgeon and R. M. Mersereau, *Multidimensional Digital Signal Processing* (Prentice-Hall, 1984).
7. Yu. V. Gulyaev, V. F. Kravchenko, and V. I. Pustovoit, Dokl. Akad. Nauk **413** (3), 320 (2007).
8. Yu. V. Gulyaev, V. F. Kravchenko, and V. I. Pustovoit, Dokl. Akad. Nauk **417** (4), 476 (2007).
9. V. F. Kravchenko and V. I. Pustovoit, Dokl. Akad. Nauk **428** (5), 601 (2009).
10. V. F. Kravchenko, V. I. Pustovoit, and D. V. Churikov, Dokl. Akad. Nauk **436** (5), 615 (2011).
11. V. F. Kravchenko, D. V. Churikov, in Program Int. Conf. "Days of Diffraction". St. Petersburg, 2011, pp. 64–65.

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