ISSN 1028-3358, Doklady Physics, 2007, Vol. 52, No. 12, pp. 645–652. © Pleiades Publishing, Ltd., 2007.
Original Russian Text © Yu.V. Gulyaev, V.F. Kravchenko, V.I. Pustovoĭt, 2007, published in Doklady Akademii Nauk, 2007, Vol. 417, No. 4, pp. 476–482.

PHYSICS

Kravchenko–Kotel'nikov Analytical Wavelets in Digital Signal Processing

Academician Yu. V. Gulyaev[a], V. F. Kravchenko[a], and Academician V. I. Pustovoĭt[b]

Received July 25, 2007

PACS numbers: 02.60.-x
DOI: 10.1134/S1028335807120026

INTRODUCTION

In this paper, a novel class of Kravchenko–Kotel'nikov analytical wavelets based on both the atomic-function (AF) theory [1] and results and concepts of [1–4] is proposed and substantiated. The efficiency of new wavelet bases is demonstrated for the analysis of ultra wideband (UWB) digital signal processing as an example.

WAVELET TRANSFORMATION

The direct continuous wavelet transformation of a one-dimensional signal $f(t)$ [1–4] is written out in the form

$$Wf(a, b) = |a|^{-1/2}\int_{-\infty}^{\infty} f(t)\tilde{\psi}\left(\frac{t-b}{a}\right)dt, \qquad (1)$$

where the symbol "~" implies complex conjugation, $\psi(t)$ is the wavelet-forming function, a is the scaling parameter, and b is the shift parameter.

GENERALIZED KOTEL'NIKOV SERIES BASED ON ATOMIC FUNCTIONS

It is well known [1] that atomic functions are widely used in various physical applications. On this basis, the generalized Kotel'nikov series are constructed [1]. Theorems determining the conditions for which a signal $s(t)$ can be accurately reconstructed according to discrete readings are called reading theorems [5, 8]. They deal with signal representations $s(t)$ in the form

$$s(t) \underset{t \in \{T\}}{\approx} \sum_k f_k(s)\varphi_k(t) = s_k(t), \qquad (2)$$

where $\varphi_k(t)$, $k = 1, 2, \ldots$, is the set of signals (functions of time) chosen a priori and independently of the form of $s(t)$ and $\{T\}$ is the set of instants of time for which $s(t)$ and $\{\varphi_k(t)\}$, $k = 1, 2, \ldots$, are defined. The sign "≈" indicates that the right-hand side of (2) does not necessarily coincide with $s(t)$ at all points but is close to $s(t)$ only in accordance with the chosen metric (distance) $\rho[s(t), s_k(t)]$.

The quadratic metric of the form

$$\rho_{s\psi} = \frac{1}{2T}\int_{-T}^{T} |s(t) - s_k(t)|^2 dt \qquad (3)$$

is the most often propagated [9]. It is proportional to the difference between the energies of the approximated signal $s(t)$ and the approximating signal $s_k(t)$. The pulses shifted proportionally to k with respect to the time reference point are usually used for the functions $\{\varphi_k(t)\}$. The pulse $\{\varphi_k(t)\}$ is called interpolating if

$$(\forall k)[\varphi_k(0) = 1], (\forall l = 1, 2, \ldots)[\varphi_k(l\Delta) = 0], \qquad (4)$$

where Δ is a certain finite time interval called the reading period. In most cases, to make the realization more convenient, we choose for the functions $f_k(t)$ the point readings

$$f_k(t) = s(k\Delta), \quad k = 0, \pm 1, \pm 2, \ldots, \qquad (5)$$

and the shape of the pulses $\{\varphi_k(t)\}$ is assumed to be identical. Then,

$$s(t) \approx \sum_k s(k\Delta)\varphi_k(t - k\Delta). \qquad (6)$$

[a] *Institute of Radio Engineering and Electronics, Russian Academy of Sciences, Building 7, ul. Mokhovaya 11, GSP-9, Moscow K-9, 125009 Russia*
[b] *Scientific and Technological Center of Unique Instrumentation, Russian Academy of Sciences, ul. Butlerova 15, Moscow, 117342 Russia*
e-mail: olegk@lianet.ru

Representation (6) yields an exact equality when $t = k\Delta$, and, within intervals between the readings, $s(t)$ varies not too rapidly. In this case, the error

$$\rho\left(s(t), \sum_k s(k\Delta)\varphi(t - k\Delta)\right) \tag{7}$$

turns out to be admissible. The stronger statement is valid for signals having a limited frequency band. For them, the spectrum

$$\hat{s}(\omega) = 0 \tag{8}$$

lies outside of the interval $|\omega| \le \Omega$, where $\Omega = 2\pi F$ is the maximum possible circular spectral frequency. Within the interval $|\omega| \le \Omega$, the spectrum $\hat{s}(\omega)$ differs from zero at least for a certain subset of the points ω of the nonzero measure (a domain at points of which $\hat{s}(\omega) \ne 0$ has a nonzero length). The spectrum of the signal $s(t)$ is determined as

$$\hat{s}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} s(t)e^{i\omega t}dt. \tag{9}$$

Here, $\hat{s}(\omega)$ is integrated in the square $[-\Omega; +\Omega]$. We suppose that the signal

$$s(t) = \int_{-\Omega}^{+\Omega} \hat{s}(\omega)e^{-i\omega t}d\omega \tag{10}$$

has a finite energy in the domain of its definition from $-\infty < t < +\infty$:

$$\int_{-\Omega}^{+\Omega} |\hat{s}(\omega)|^2 d\omega < \infty. \tag{11}$$

We now represent the signal [1, 8] with the limited band $\hat{s}(\omega)$ in the form

$$s(t) = \sum_{k=-\infty}^{\infty} s(k\Delta)\prod_{j=1}^{\infty} \mathrm{sinc}\left(\frac{\pi}{\Delta a^{j-1}}(t - k\Delta)\right). \tag{12}$$

Expression (12) satisfies all requirements of the Kotel'nikov theorem and has a better convergence, especially while reconstructing signals localized in time and disconnected signals. In calculations, it is necessary to restrict them by a finite number of product terms in the right-hand side of (12). In this case, the exact expansion

$$s(t) = \sum_{k=-\infty}^{\infty} s(k\Delta)\prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi}{\Delta a^{j-1}}(t - k\Delta)\right), \tag{13}$$

$$a(1 + a^{-M}) > 2, \quad \Delta = \frac{\pi}{\Omega}\frac{a(1 + a^{-M}) - 2}{a - 1} \tag{14}$$

takes place.

The minimum values of the parameter a can be found from the solution of the transcendent equation $a(1 + a^{-M}) = 2$. For $M = 1$, the Kotel'nikov series [1] follows from (13). In the limiting case when $M \to \infty$, we arrive at series (12).

THE PHYSICAL SENSE OF THE GENERALIZED KOTEL'NIKOV SERIES

Expression (12) valid under condition (14) testifies to the fact that the discretization operations (digital transformation) and the signal reconstruction are mutually inverse. It follows from (12) that, for any arbitrarily given t, the signal $s(t)$ can be obtained at the receiving end theoretically. To do this, we should transmit the entire infinite sequence $s(k\Delta)$, $k = \ldots, -2, 0, +1, +2, \ldots$, through an ideal low-frequency filter having the cut frequency $\Omega = \frac{\pi}{\Delta}\frac{a(1 + a^{-M}) - 2}{a - 1}$ and the amplitude–frequency characteristic $|K(i\omega)| = \Delta$ within its transmission band. The Kotel'nikov theorem characterizes the ultimate potentialities of the communication channel. Then, we should understand relationship (13) as

$$s(t) = \lim_{K \to \infty}\left[\sum_{k=-K}^{K} s(k\Delta)\prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi}{\Delta a^{j-1}}(t - k\Delta)\right)\right],$$
$$s_k(t; \Delta) = \sum_{k=-K}^{K} s(k\Delta)\prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi}{\Delta a^{j-1}}(t - k\Delta)\right). \tag{15}$$

The convergence is determined in the quadratic metric [9]

$$\rho(s(t); \tilde{s}(t; \Delta)) = \int_{-\infty}^{\infty} |s(t) - \tilde{s}(t; \Delta)|^2 dt, \tag{16}$$

where $\tilde{s}(t; \Delta) = \lim_{k \to \infty} s_k(t; \Delta)$.

THE KRAVCHENKO–KOTEL'NIKOV WEIGHT FUNCTION (WINDOW)

In accordance with (6), we can rewrite Eq. (13) in the form

$$s(t) = \sum_{k=-\infty}^{\infty} s(k\Delta)\varphi_{k-M}(t), \tag{17}$$

where

$$\varphi_{k-M}(t) = \prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi}{\Delta a^{j-1}}(t - k\Delta)\right). \tag{18}$$

The parameter a, $1 < a \le 2$, is chosen according to the frequency properties of signals [1]. Function (18) possesses good physical parameters. The Kravchenko–

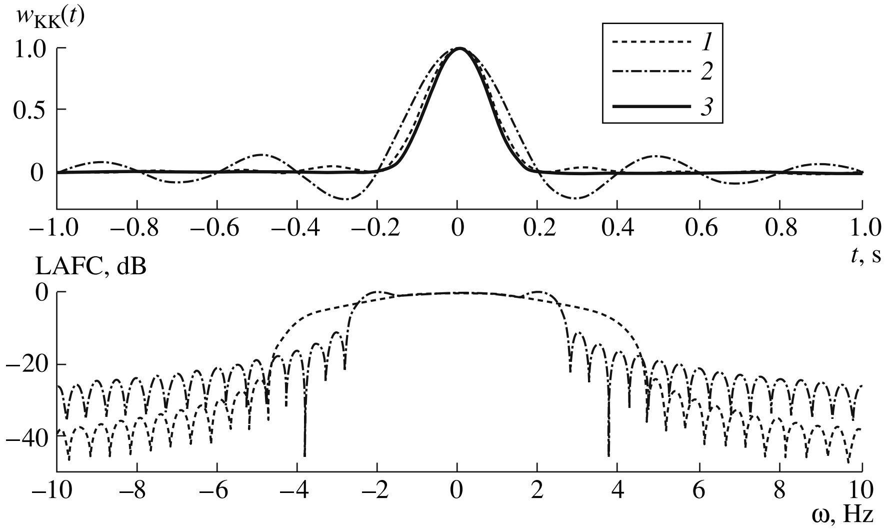

Fig. 1. Kravchenko–Kotel'nikov weight functions $w_{KK}(t)$ for *1*, $M = 1$; *2*, 2; *3*, 3 ($a = 1.2$, $\Delta = 0.2$ s) and their logarithmic amplitude-frequency characteristics (LAFC).

Kotel'nikov weight function (window) is determined by the expression

$$w_{KK}(t) = \varphi_{0-M}(t) = \prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi}{\Delta a^{j-1}} t\right). \quad (19)$$

Three parameters, namely, M, a, and Δ, affect the behavior of this function. Changing them, we obtain the Kravchenko–Kotel'nikov weight functions with different physical characteristics. Figure 1 exhibits the Kravchenko–Kotel'nikov weight functions for different values of M, a, and Δ.

THE SECOND DERIVATIVE OF THE KRAVCHENKO–KOTEL'NIKOV WEIGHT FUNCTION

We now consider the behavior of the Kravchenko–Kotel'nikov weight function (window)

$$w''_{KK}(t) = -\frac{d^2}{dt^2}\left(\prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi}{\Delta a^{j-1}} t\right)\right). \quad (20)$$

The behavior of this function for different values of the parameters M, a, and Δ is shown in Fig. 2.

WAVELET-MODULATED KRAVCHENKO–KOTEL'NIKOV WEIGHT FUNCTIONS

In accordance with [3, 4], the wavelet modulated weight function is determined by the expression of the form

$$\psi(t) = w(t)\exp(i\eta t), \quad (21)$$

where $w(t)$ is the weight function and η is the parameter whose physical sense is frequency. The weight function satisfies the normalization conditions $w(t) = 0$ for $|t| > 1$, $w(0) = 1$, $w(-t) = w(t)$. In the case of constructing the analytical wavelet, the basic requirement on this function is a sufficiently good localization in both the time region and the frequency region. If $\eta = \pi m$, where m is a natural number, then m characterizes the number of total periods of a harmonic function, which are cut by the window $w(t)$. Varying the parameter η, we can attain the fulfillment of the condition $\hat{\psi}(\omega) = 0$ for all $\omega \le 0$. Transformation (21) does not affect the carrier width in the time region. Thus, we arrive at the analytical Kravchenko–Kotel'nikov wavelets

$$\psi_{AKK}(t) = w_{KK}(t)\exp(i\eta t) = \prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi t}{\Delta a^{j-1}}\right)\exp(i\eta t). \quad (22)$$

The parameters M, a, and Δ entering into (22) determine the basic physical characteristics (the width of the spectral-density function at the levels of 3 and 6 dB, the maximum level of side lobes, etc.). These functions are plotted in Fig. 3. The second derivative of expression (20) is of a certain physical interest. This derivative is of the form

$$\psi_{AKK}(t) = w''_{KK}(t)\exp(i\eta t) = \left(-\frac{d^2}{dt^2}\prod_{j=1}^{M} \mathrm{sinc}\left(\frac{\pi t}{\Delta a^{j-1}}\right)\right)\exp(i\eta t). \quad (23)$$

The plot of function (23) is given in Fig. 3.

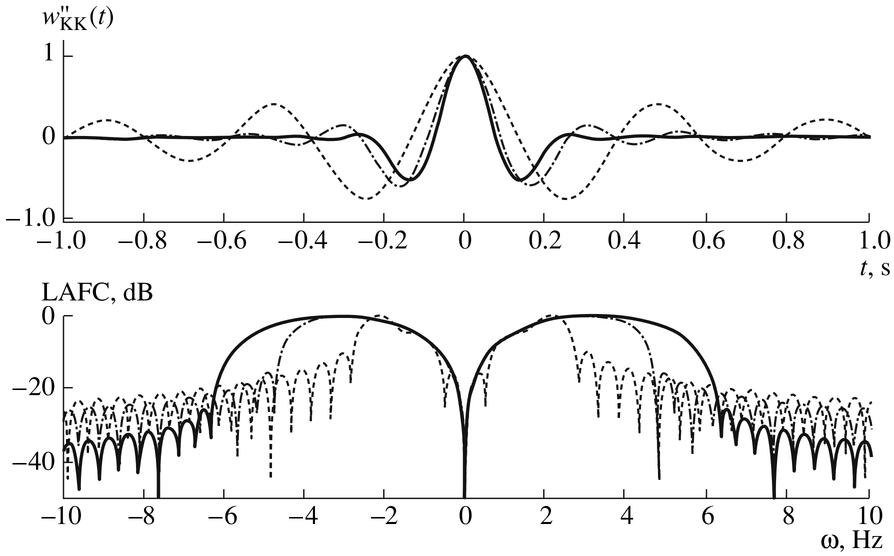

Fig. 2. Second derivative $w_{KK}(t)$ of the Kravchenko–Kotel'nikov weight functions for $M = 1$, 2, and 3, $a = 1.2$, $\Delta = 0.2$ s and their logarithmic amplitude-frequency characteristics (LAFC). Denotations are the same as in Fig. 1.

A MODEL FOR ULTRA WIDEBAND SIGNALS

We now consider a model for the following UWB signals [6, 7]:

$$\text{(i) } y_1(t) = -\left(\operatorname{step}\left(\frac{t+0.5}{\tau}\right) - \operatorname{step}\left(\frac{t-0.5}{\tau}\right)\right)\operatorname{sgn}(t);$$

$$\text{(ii) } y_2(t) = (-1)^n \sin\left(\pi n \frac{t}{\tau}\right)\exp\left(-\left|\frac{t}{\tau}\right|\right) \times\left(\operatorname{step}\left(\frac{t}{\tau}+1\right) - \operatorname{step}\left(\frac{t}{\tau}-1\right)\right);$$

$$\text{(iii) } y_3(t) = -\frac{2t}{\tau^2}\exp\left(-\left(\frac{t}{\tau}\right)\right)^2;$$

$$\text{(iv) } y_4(t) = -\frac{2}{\tau^2}\left(1 - \frac{2t^2}{\tau^2}\right)\exp\left(-\left(\frac{t}{\tau}\right)^2\right);$$

$$\text{(v) } y_5(t) = \exp\left(-\left(\frac{t}{2\tau}\right)^2\right)n!\sum_{k=0}^{[n/2]}\left(-\frac{1}{2}\right)^k \frac{(t/\tau)^{n-2k}}{(n-2k)!k!},$$

where $\operatorname{step}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}$ is the Heaviside function and $\operatorname{sgn} t$ is the function characterizing the sign of the argument. The physical characteristics of these functions are presented in Table 1.

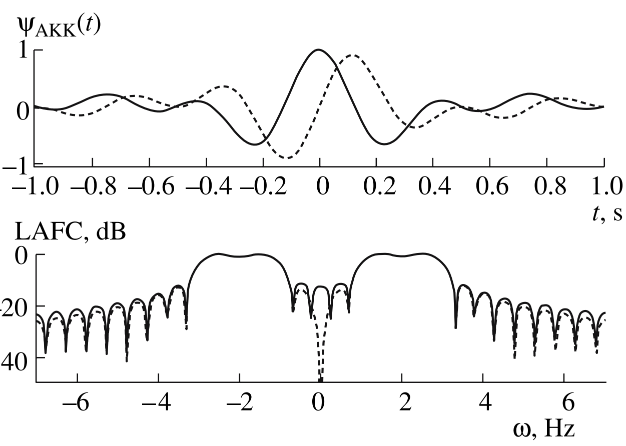

Fig. 3. Wavelet-modulated Kravchenko–Kotel'nikov weight function (constructed for $\mu = 1$) and its logarithmic amplitude-frequency characteristic (LAFC). $M = 1$, $a = 2$, $\Delta = 0.5$ s. Solid and dashed lines correspond to the real and imaginary parts, respectively.

PHYSICAL CHARACTERISTICS OF ULTRA WIDEBAND SIGNALS

In order to study wavelet-forming functions for the model UWB signals to be analyzed, we make use of modified physical characteristics invariant with respect to variations of the parameter a [see (1)] [1, 4]: the broadband index μ; the central frequency f_0 of the spectral-density function (SDF); the relative position γ_1 of the SDF maximum; the relative position $\gamma_2 = \frac{\nu - f_m}{f_{\max} - f_{\min}}$ of the first SDF zero; the relative width $\gamma_3 = \frac{f_{\max 3} - f_{\min 3}}{f_{\max} - f_{\min}}$ of the SDF at the 3-dB level; the relative

Table 1. Model ultra wideband (UWB) signals and their physical characteristics

Ordering number of realization	Model UWB signals	μ	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9
1	$y_1(t)$, $\tau = 1$ s	1.54	0.91	1.04	0.85	1.11	–1.58	–1.41	10.01	10.01	–5.29
2	$y_1(t)$, $\tau = 1.5$ s	1.50	0.92	1.00	0.83	1.11	–1.49	–1.34	6.67	6.67	–5.28
3	$y_2(t)$, $n = 1$	1.46	0.91	1.06	0.81	1.13	–11.47	–7.81	8.06	6.28	–12.99
4	$y_2(t)$, $n = 2$	0.79	0.98	0.94	0.88	1.18	–10.18	–6.32	9.84	6.57	–8.19
5	$y_2(t)$, $n = 3$	0.52	1.02	0.88	0.82	1.18	–9.53	–5.53	10.60	6.63	–8.60
6	$y_3(t)$, $\tau = 0.3$ s	1.59	0.87	2.26	0.81	1.13	–37.58	–9.17	14.30	10.44	–49.15
7	$y_3(t)$, $\tau = 0.5$ s	1.48	0.87	1.82	0.88	1.18	–8.80	–5.58	8.74	6.49	–18.87
8	$y_4(t)$, $\tau = 0.3$ s	1.22	0.90	2.19	0.84	1.16	–32.05	–8.43	19.43	10.65	–43.44
9	$y_4(t)$, $\tau = 0.5$ s	1.20	1.00	1.11	0.83	1.17	–6.20	–4.60	12.18	6.93	–14.08
10	$y_5(t)$, $n = 1$, $\tau = 0.15$ s	1.59	0.87	2.26	0.81	1.13	–37.58	–9.17	14.30	10.44	–49.15
11	$y_5(t)$, $n = 3$, $\tau = 0.15$ s	1.77	0.85	1.39	0.89	1.09	–25.77	–10.44	11.50	8.12	–34.63
12	$y_5(t)$, $n = 5$, $\tau = 0.15$ s	1.81	0.84	1.08	0.93	1.07	–17.01	–10.40	10.20	6.98	–26.63
13	$y_5(t)$, $n = 1$, $\tau = 0.1$ s	1.59	0.86	3.28	0.85	1.15	–96.77	–8.86	21.44	15.67	–108.30
14	$y_5(t)$, $n = 3$, $\tau = 0.1$ s	1.75	0.82	2.21	0.90	1.10	–81.50	–10.37	17.25	12.17	–92.79
15	$y_5(t)$, $n = 5$, $\tau = 0.1$ s	1.83	0.85	1.69	0.93	1.06	–68.61	–11.34	15.27	10.43	–77.97
16	$y_5(t)$, $n = 7$, $\tau = 0.1$ s	1.85	0.87	1.42	0.95	1.06	–57.50	–11.65	14.02	9.32	–67.70

width $\gamma_4 = \dfrac{f_{\max 6} - f_{\min 6}}{f_{\max} - f_{\min}}$ of the SDF at the 6-dB level; the information loss γ_5 (expressed in dB) observed in the case of ignoring the SDF components at the frequencies $f \notin [0, \nu]$; the information loss γ_6 (expressed in dB) observed in the case of ignoring the SDF components at the frequencies $f \notin [f_{\min}, f_{\max}]$. The coherent amplification, the equivalent noise band, and the maximum level of side lobes (expressed in dB) are given by the quantities γ_7, γ_8, and γ_9, respectively.

THE QUALITY FUNCTIONAL FOR THE CHOICE OF THE WAVELET BASIS IN THE ANALYSIS OF SUPER-BROADBAND PULSES

For analysis of UWB signals, a quality functional was introduced in [1, 4], which allows us to estimate the choice of basis wavelet functions. The functional is of the form

$$J(\psi, y) = \sum_{k=0}^{N} \left| \frac{\gamma_k^{\psi} - \gamma_k^{y}}{\gamma_k^{y}} \right|^2,$$

where $\psi(t)$ is the wavelet-forming function, $y(t)$ is the signal to be analyzed, γ_k^{ψ} and γ_k^{y} are their physical characteristics, and N is the number of parameters to be compared. Here, $\gamma_0 = \mu$ and $N = 4$. When the wavelet basis is being chosen, it is reasonable to allow for a number of frequency characteristics (23) invariant with respect to the scaling of the argument. The optimal choice of the basis corresponds to the minimum value of the quality functional. The analysis of the expressions for calculating both the physical parameters and the quality functional has shown that the optimal results are attained in the case when the signal wavelet function has close broadband indices: $\mu^{\psi} \approx \mu^{y}$. Insofar as the spectrum of the Kravchenko–Kotel’nikov function is symmetric [1, 8], in accordance with (22), for the wavelet-modulated functions (20), (23), relationships $\mu^{\psi} = f_{\max}^{w} \dfrac{2}{\eta}$ are valid. Here, $f_{\max}^{w}$ is the maximum frequency of the SDF $\hat{w}(\omega)$ and $\eta = f_{\max}^{w} \dfrac{2}{\mu^{\psi}}$. In practice, the physical parameters are determined separately for the real and imaginary parts; therefore, $\eta \approx f_{\max}^{w} \dfrac{2}{\mu^{\psi}}$. Then, (21) takes the form $\psi(t) = w(t)\exp\left(i\dfrac{2f_{\max}^{w}}{\mu'}t\right)$. Here, μ' is the given broadband index. The function $\psi(t)$ is the wavelet

Table 2. Physical characteristics of weight functions (windows) and of Kravchenko–Kotel'nikov wavelets

Weight functions (windows)	M	a	Δ	η	μ	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9
$w_{KK}(t)$	1	1.2	0.20	5.00	2.00	0.00	1.13	0.98	1.04	–14.79	–12.61	27.40	14.71	–11.37
$Re(\psi_{AKK}(t))$	1	1.2	0.20	5.00	1.02	1.00	0.55	0.96	1.02	–17.16	–13.32	43.93	18.90	–11.05
$Im(\psi_{AKK}(t))$	1	1.2	0.20	5.00	1.00	1.00	0.56	0.98	1.04	–18.40	–13.55	42.09	17.36	–11.47
$w_{KK}(t)$	2	1.2	0.20	6.00	2.00	0.00	1.58	0.82	1.17	–28.55	–9.30	48.16	33.50	–24.20
$Re(\psi_{AKK}(t))$	2	1.2	0.20	6.00	1.00	1.00	0.79	0.82	1.17	–31.07	–9.27	75.74	41.43	–24.00
$Im(\psi_{AKK}(t))$	2	1.2	0.20	6.00	1.00	1.00	0.79	0.82	1.17	–32.02	–9.28	75.56	41.23	–24.24
$w_{KK}(t)$	3	1.2	0.20	6.45	2.00	0.00	1.96	0.85	1.17	–40.60	–8.78	56.22	40.70	–36.12
$Re(\psi_{AKK}(t))$	3	1.2	0.20	6.45	1.00	1.00	0.98	0.85	1.17	–43.11	–8.74	88.30	50.21	–33.11
$Im(\psi_{AKK}(t))$	3	1.2	0.20	6.45	1.00	1.00	0.98	0.85	1.17	–43.95	–8.75	88.31	50.21	–36.17
$w_{KK}(t)$	1	1.5	0.20	5.00	2.00	0.00	1.13	0.98	1.04	–14.79	–12.61	27.40	14.71	–11.37
$Re(\psi_{AKK}(t))$	1	1.5	0.20	5.00	1.02	1.00	0.55	0.96	1.02	–17.16	–13.32	43.93	18.90	–11.05
$Im(\psi_{AKK}(t))$	1	1.5	0.20	5.00	1.00	1.00	0.56	0.98	1.04	–18.40	–13.55	42.09	17.36	–11.47
$w_{KK}(t)$	2	1.5	0.20	5.82	2.00	0.00	1.48	0.84	1.14	–26.56	–10.30	44.70	31.08	–22.96
$Re(\psi_{AKK}(t))$	2	1.5	0.20	5.82	1.00	1.00	0.74	0.84	1.14	–29.89	–10.29	70.03	38.14	–23.00
$Im(\psi_{AKK}(t))$	2	1.5	0.20	5.82	1.00	1.00	0.74	0.84	1.14	–29.04	–10.27	70.43	38.58	–22.68
$w_{KK}(t)$	3	1.5	0.20	5.82	2.00	0.00	1.81	0.86	1.14	–37.41	–9.37	49.20	35.58	–33.55
$Re(\psi_{AKK}(t))$	3	1.5	0.20	5.82	1.00	1.00	0.91	0.86	1.14	–40.81	–9.34	77.25	43.86	–33.10
$Im(\psi_{AKK}(t))$	3	1.5	0.20	5.82	1.00	1.00	0.91	0.86	1.14	–39.83	–9.34	77.34	43.96	–33.50
$w_{KK}(t)$	1	2.0	0.20	5.00	2.00	0.00	1.13	0.98	1.04	–14.79	–12.61	27.40	14.71	–11.37
$Re(\psi_{AKK}(t))$	1	2.0	0.20	5.00	1.02	1.00	0.55	0.96	1.02	–17.16	–13.32	43.93	18.90	–11.05
$Im(\psi_{AKK}(t))$	1	2.0	0.20	5.00	1.00	1.00	0.56	0.98	1.04	–18.40	–13.55	42.09	17.36	–11.47
$w_{KK}(t)$	2	2.0	0.20	5.63	2.00	0.00	1.39	0.87	1.10	–24.84	–11.59	40.36	27.15	–21.50
$Re(\psi_{AKK}(t))$	2	2.0	0.20	5.63	1.00	1.00	0.69	0.87	1.10	–27.78	–11.59	63.54	33.64	–21.34
$Im(\psi_{AKK}(t))$	2	2.0	0.20	5.63	1.00	1.00	0.69	0.87	1.10	–27.56	–11.60	63.31	33.40	–21.47
$w_{KK}(t)$	3	2.0	0.20	5.63	2.00	0.00	1.55	0.87	1.10	–32.65	–11.16	42.71	29.72	–29.37
$Re(\psi_{AKK}(t))$	3	2.0	0.20	5.63	1.00	1.00	0.77	0.87	1.10	–35.56	–11.13	67.24	36.83	–29.28
$Im(\psi_{AKK}(t))$	3	2.0	0.20	5.63	1.00	1.00	0.78	0.87	1.10	–35.40	–11.13	66.98	36.54	–29.33

provided that the condition of the zero average is fulfilled. The expression for $\psi(t)$ is reasonable to apply when $\frac{2f^w_{max}}{\mu'} \geq f^w_{15}$, where $10\log[\hat{w}(f^w_{10})] = -15$ dB. This is determined by both the requirements on the accuracy of the validity of the zero-average condition and the frequency properties. If $\frac{2f^w_{max}}{\mu'} < f^w_{10}$, then $\eta = f_{10}$ and $\psi(t) = w(t)\exp(i f^w_{10} t)$. Thus, we arrive at the wavelet-modulated weight (window) functions consistent with UWB signals via the parameter μ.

CONCLUSIONS

Thus, we have performed a numerical experiment for both model ultra wideband signals and their physical characteristics (Table 1), as well as for the weight functions (windows), and for the Kravchenko–Kotel'nikov wavelet-modulated functions (Tables 2, 3). In Table 4, values of the quality functional are presented for model UWB signals and for the Kravchenko–Kotel'nikov wavelets. The Kravchenko–Kotel'nikov analytical wavelet transformation extends the potentialities of the continuous wavelet transformation for the UWB-signal analysis. This is explained by the fact that instead of one physical characteristic

Table 3. Physical characteristics of weight functions (windows) and of wavelets constructed for the second derivative of the Kravchenko–Kotel'nikov function

Weight functions (windows)	M	a	Δ	η	μ	γ_1	γ_2	γ_3	γ_4	γ_5	γ_6	γ_7	γ_8	γ_9
$w_{2_d\mathrm{KK}}(t)$	1	1.2	0.20	5.18	0.53	1.04	0.63	0.75	1.50	–8.94	–4.80	15.70	8.16	–10.21
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	1	1.2	0.20	5.18	1.00	1.00	0.54	0.96	1.02	–12.05	–9.16	24.82	10.19	–10.17
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	1	1.2	0.20	5.18	1.00	1.00	0.54	0.96	1.02	–10.88	–8.64	24.51	9.94	–10.21
$w_{2_d\mathrm{KK}}(t)$	2	1.2	0.20	8.54	1.08	1.08	0.58	0.86	1.12	–15.56	–9.44	34.58	17.90	–16.11
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	2	1.2	0.20	8.54	1.00	1.00	0.55	0.97	1.02	–16.80	–13.22	53.56	21.47	–16.06
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	2	1.2	0.20	8.54	1.00	1.00	0.55	0.97	1.02	–20.35	–14.20	55.07	22.69	–15.89
$w_{2_d\mathrm{KK}}(t)$	3	1.2	0.20	10.27	1.18	0.99	0.82	0.86	1.13	–26.93	–9.55	49.11	27.83	–26.28
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	3	1.2	0.20	10.27	1.00	1.00	0.62	0.95	1.04	–32.69	–13.15	76.32	33.61	–26.24
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	3	1.2	0.20	10.27	1.00	1.00	0.62	0.95	1.04	–28.19	–13.07	77.97	35.08	–26.24
$w_{2_d\mathrm{KK}}(t)$	1	1.5	0.20	5.18	0.53	1.04	0.63	0.75	1.50	–8.94	–4.80	15.70	8.16	–10.21
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	1	1.5	0.20	5.18	1.00	1.00	0.54	0.96	1.02	–12.05	–9.16	24.82	10.19	–10.17
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	1	1.5	0.20	5.18	1.00	1.00	0.54	0.96	1.02	–10.88	–8.64	24.51	9.94	–10.21
$w_{2_d\mathrm{KK}}(t)$	2	1.5	0.20	7.72	1.06	1.06	0.61	0.88	1.12	–17.21	–9.52	31.63	16.48	–15.90
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	2	1.5	0.20	7.72	1.00	1.00	0.56	0.98	1.02	–21.87	–14.28	49.03	19.80	–15.97
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	2	1.5	0.20	7.72	1.00	1.00	0.56	0.98	1.02	–18.77	–13.43	50.32	20.86	–15.70
$w_{2_d\mathrm{KK}}(t)$	3	1.5	0.20	8.54	1.13	1.02	0.81	0.84	1.15	–24.77	–9.59	40.41	22.91	–24.60
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	3	1.5	0.20	8.54	1.00	1.00	0.62	0.95	1.05	–26.10	–12.62	62.41	27.33	–24.59
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	3	1.5	0.20	8.54	1.00	1.00	0.62	0.95	1.05	–29.55	–12.71	64.50	29.19	–24.48
$w_{2_d\mathrm{KK}}(t)$	1	2.0	0.20	5.18	0.53	1.04	0.63	0.75	1.50	–8.94	–4.80	15.70	8.16	–10.21
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	1	2.0	0.20	5.18	1.00	1.00	0.54	0.96	1.02	–12.05	–9.16	24.82	10.19	–10.17
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	1	2.0	0.20	5.18	1.00	1.00	0.54	0.96	1.02	–10.88	–8.64	24.51	9.94	–10.21
$w_{2_d\mathrm{KK}}(t)$	2	2.0	0.20	6.91	1.04	1.10	0.60	0.88	1.15	–16.47	–9.39	28.39	15.03	–15.34
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	2	2.0	0.20	6.91	1.00	1.00	0.56	0.97	1.04	–18.74	–12.99	44.04	18.08	–15.34
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	2	2.0	0.20	6.91	1.00	1.00	0.57	0.97	1.04	–19.28	–13.00	45.12	18.98	–15.16
$w_{2_d\mathrm{KK}}(t)$	3	2.0	0.20	7.18	1.04	1.06	0.78	0.87	1.17	–24.95	–9.23	32.52	18.19	–22.32
$\mathrm{Re}(\psi_{\mathrm{A2dKK}}(t))$	3	2.0	0.20	7.18	1.00	1.00	0.61	0.96	1.04	–28.33	–13.13	50.11	21.61	–22.37
$\mathrm{Im}(\psi_{\mathrm{A2dKK}}(t))$	3	2.0	0.20	7.18	1.00	1.00	0.61	0.96	1.04	–27.26	–13.08	52.00	23.26	–22.21

Table 4. The values of the quality functional for model ultra wideband signals (1–16) and Kravchenko–Kotel'nikov wavelets

Weight-function parameters			1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
M	a	Δ																
1	1.2	0.2	24.47	23.35	27.77	19.24	18.41	64.19	52.82	61.06	28.64	64.19	39.16	26.01	76.13	60.99	47.09	38.24
2	1.2	0.2	8.45	6.47	8.24	3.25	1.17	46.76	36.67	42.77	8.50	46.76	28.66	19.24	64.32	52.38	40.51	31.97
3	1.2	0.2	6.61	5.34	5.16	0.44	1.28	40.83	27.60	31.89	1.44	40.83	23.53	17.70	60.10	46.77	34.69	26.76
1	1.5	0.2	24.47	23.35	27.77	19.24	18.41	64.19	52.82	61.06	28.64	64.19	39.16	26.01	76.13	60.99	47.09	38.24
2	1.5	0.2	9.50	7.73	10.27	4.81	2.76	48.36	37.77	45.02	10.89	48.36	29.10	18.80	65.11	52.67	40.45	31.80
3	1.5	0.2	5.74	4.28	5.17	0.49	0.29	42.56	29.75	35.54	3.76	42.56	23.88	16.11	61.07	47.61	35.09	26.73
1	2.0	0.2	24.47	23.35	27.77	19.24	18.41	64.19	52.82	61.06	28.64	64.19	39.16	26.01	76.13	60.99	47.09	38.24
2	2.0	0.2	11.92	10.79	13.26	7.91	5.39	51.03	41.12	48.40	14.32	51.03	30.33	19.15	66.88	53.63	40.92	32.19
3	2.0	0.2	7.49	6.00	8.62	3.50	2.20	46.51	35.45	43.87	9.26	46.51	25.96	15.66	63.52	49.76	36.85	28.14

(|$W(a, b)$|), we can simultaneously analyze four: (|$W(a, b)$|), arg $W(a, b)$, Re $W(a, b)$ and Im $W(a, b)$. We have also shown that the Kravchenko–Kotel'nikov wavelets can be efficiently used in solving problems associated with the digital processing of UWB signals. We have proposed and substantiated a method that allows us to make analytical wavelets consistent with particular UWB signals, which improves the quality of both the analysis and digital signal processing. Estimating the efficiency of the Kravchenko–Kotel'nikov wavelet bases, the quality functional makes it possible to find a basis suitable for practice among the set of existing ones. The numerical experiment carried out and the analysis of the physical results obtained show high potentialities for the application of the novel class of analytical wavelets in solving various problems in the fields of radio physics, radio location, radio vision, and acoustical optics.

REFERENCES

1. V. F. Kravchenko and V. L. Rvachev, *Boolean Algebra, Atomic Functions, and Wavelets in Physical Applications* (Fizmatlit, Moscow, 2006) [in Russian].
2. Yu. V. Gulyaev, V. F. Kravchenko, and V. I. Pustovoĭt, Dokl. Akad. Nauk **413**, 320 (2007).
3. S. G. A. Mallat, *A Wavelet Tour of Signal Processing* (Academic, New York, 1998).
4. V. F. Kravchenko, O. V. Lazorenko, and L. F. Chernogor, Usp. Sovrem. Radioélektron., No. 5, 29 (2007).
5. A. A. Kharkevich, *Spectra and Analysis* (LKI, Moscow, 2007) [in Russian].
6. A. N. Denisenko, *Handbook of Signals and Theoretical Radio Engineering* (Goryachaya Liniya-Telekom, Moscow, 2005) [in Russian].
7. M. Ghavami, L. B. Michael, and R. Kohno, *Ultra Wideband Signals and Systems in Communication Engineering* (New York, Wiley, 2004).
8. *Digital Signal and Image Processing in Radiophysical Applications*, Ed. by V.F. Kravchenko (Fizmatlit, Moscow, 2007).
9. N. I. Akhiezer, *Lectures on the Approximation Theory*, (Nauka, Moscow, 1965).

Translated by G. Merzon